CAPITAL BUDGETING DECISIONS WITH

POST-AUDIT INFORMATION

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DISSERTATION ABSTRACT

CAPITAL BUDGETING DECISIONS WITH

POST-AUDIT INFORMATION

George Chikajiro Prueitt

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The purpose of this research is to investigate the significance of using post-audit information in capital budgeting decisions. The focus is directed at implemented projects whose proposals included cash flow estimates that were represented as probability functions, and the subsequent cash flow realizations of those projects. Since the cash flows must be recorded for other reporting purposes, they represent an essentially "free" source of information. Bayes theorem is used to consider this post-audit information.

The Bayesian techniques consider the probabilistic cash flows when they are modeled as one of three general situations: (1) discrete functions, (2) continuous functions (beta or normal probability distributions), or (3) discrete approximations to continuous functions. This research introduces a concept that permits incorporation of the user's strength of beliefs, or confidence, in the

quality of the estimates that were used during project selection. In similar fashion, this research introduces the equivalent sample size concept that permits the user to incorporate his strength of belief in the quality, or replicability, of the sample information. These concepts are then applied to case studies of actual decision problems. While the decision problems specifically concern equipment replacement, the concepts that are developed can be generalized for other capital budgeting situations.

As the revision procedures generate the uncertainty resolution for the project cash flows, it is helpful to provide the user with a graphic illustration of this resolution. For this purpose, Cash Flow Control Charts are developed as a management tool. These charts have appearances that are similar to statistical quality control charts, but they incorporate Bayesian revision procedures.

Finally, the research concludes that incorporation of post-audit information can lead to changes in company investment strategies. The methods led to improved decision making in the timing and magnitude of equipment investments.

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This research investigates using post-audit information in capital budgeting decisions. The effects of cash flow realizations, incorporated by Bayes theorem, on cash flow estimates represented as probability functions are examined. This research develops modeling techniques, and concepts that incorporate the user's strength of belief in his input information and in the available sample information. These concepts are then applied to case studies of actual decision problems. As the revision procedures generate the uncertainty resolution for the project cash flows, it is helpful to have a graphic illustration. This research develops Cash Flow Control Charts as a management tool. These charts are similar to statistical quality control charts, but incorporate Bayesian revision. The methods lead to improved decision making.

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I. INTRODUCTION

Capital budgeting decisions form one of the most critical and difficult areas of business decision making. The decisions are important because they affect the economic welfare of the company. The decisions are difficult because they usually involve unknown future events, and budgeting constraints limit how many investments the company can initiate. As these decisions become increasingly complex, businessmen need analytic tools that enhance their decision making capabilities.

The subject of capital budgeting covers a broad spectrum of theories and practices that businessmen use to maximize the wealth of their respective firms. These include (but are not limited to) the development of capital expenditure programs, identification of new investment opportunities, estimation of the future cash flows for these investments, comparison and selection of investments, and management and review of the projects within the company program. This study is concerned with the development of analytic decision making techniques that will aid the businessman during his monitoring and management of implemented projects. The techniques have impacts on the amount of capital resources available, how initial estimates can be revised through time, and how these revisions can affect the company's existing project decision strategies.

Problem Definition

The techniques that are developed in this study can be applied to many different investment situations, but they are most vividly represented by the circumstances that exist for manufacturing equipment replacement decisions when technological innovations are present.

Replacement Situations

Traditional equipment replacement analysis has focused on the question: Should we replace the present asset (called defender) with an alternative asset (called challenger), or should we keep the defender and make the replacement sometime in the future. The "asset" in question has typically been one specific piece of machinery. For this study, the scope of the equipment replacement decision is expanded from the traditional machine for machine comparison (serial replacement decisions), to encompass decisions that compare entire defender production processes to innovative challenger production processes (parallel replacement decisions). In these replacement situations, the cash flows are modeled as probability distributions (either discrete or continuous). Additionally, this study is only concerned with replacement problems that have capital budget limits, which will limit the number, or type, of alternatives that are feasible for consideration.

When an equipment replacement alternative involves a technological innovation, it frequently has higher investment costs than replacement alternatives without innovations. The increased costs require more of a company's limited budget. Therefore, these budgetary limits make it necessary for the businessman to consider invested

capital as well as other existing capital (equities or loans), as company assets. While invested capital is not a very liquid asset, it still retains an inherent salvage, or abandonment value. Therefore, it is an important company resource that must be utilized in the most efficient fashion. Once any project is initiated, it is an asset whose performance is subject to review at any subsequent point in time.

Decision Environment

Company investment decision policies can be described by two fundamental categories. In the first, the initial investment decision is the only invest/termination decision to be made over the project's economic life. In general, firms only permit this type of policy if the project has small capital requirements (under some cut-off level of investment), or if their staffs are too small to monitor all projects, or if the firm follows some very simple capital budgeting scheme.

The second policy, the one of interest, is where the firm makes regular periodic reviews of all of its investments. If a project is executed to its fully planned horizon, it is selected for retention at each review. Project termination may occur at any of the sequential decision points. Termination may be for either poor project performance or the availability of new investments that have greater benefits, effectively "bumping" existing investments.

Post-audit Information

When a project is considered for implementation, its periodic cash flows (expenditures and receipts) are represented by estimated values. At project completion, its return on investment is determined

from its actual, or realized, cash flows. These realizations are post-audit information. At any intermediate point in the project's life, an up-to-date estimate of the project's return can be found by combining the remaining estimates with the existing realizations. However, the use of post-audit information varies widely from firm to firm.

In some companies, the cash flow information is only used for accounting purposes. Expenditures are compared to their budgetary limits (with adjustments, as required, for excess expenditures), receipts are recorded as they occur, and the information is only used for required reports (stockholders, taxes, etc). In other companies, the information also goes to the project originator. He determines where and, hopefully, why his estimates deviated from the realizations. He also makes adjustments, when appropriate, in his methods, to improve the quality of his estimates in future proposals. In other companies, the information is not provided to the project originators, but to the decision makers, who use the information to form opinions about the bias their staffs may have in preparing proposals. Lastly, some firms provide this feedback to both groups, for their individual reviews.

Because the information is collected for standard reporting purposes, this post-audit information can be considered as essentially "free" data. The only costs involved are for collating the information into a usable form (as directed by the businessman conducting the review). This information does not carry the "cost" normally associated with data sampling or testing.

Uncertainty Resolution

The primary interest of this study is determining how the uncertainty attached to future cash flow estimates is resolved through the use of post-audit information (cash flow realizations), and the effects of the revisions on the company's capital budgeting strategy (which was based on the original estimates), as the company moves through a sequential decision making process. When businessmen prepare to make their decisions, they seek the most accurate information available. They take this information and formulate estimates for unknown future values. At project selection, these estimates are the best (and only) information available. Once the project is initiated and moves through its economic life, the original cash flow estimates are replaced by profit and loss realizations. Each successive realization reduces some of the original performance uncertainty, until, at the end of the project's life, actual project performance eliminates all uncertainty. This process of moving from greater uncertainty toward less uncertainty is referred to as uncertainty resolution. As this uncertainty resolution occurs in a given project, the anticipated profitability of that project may be changed to the degree that changes are needed in the company's plans for that project.

Process of Revising Estimates

The concept of using post-audit information (sample data) to revise an initial performance estimate (prior belief), to obtain a revised estimate (posterior belief) fits the classic Bayesian revision framework. While Bayesian techniques have been used in many areas (including some portfolio models), they have not been applied to

probability based models for equipment replacement decisions. This study examines some existing Bayesian revision models and extends their conceptualizations to cover several situations that occur frequently in these replacement problems. The extended concepts permit the businessman to incorporate his subjective opinions, or "strength of beliefs", in the quality of the initial estimates, and to similarly incorporate his beliefs about the relative quality of the observed cash flow results.

Problem Statement

The importance of this uncertainty resolution process is that, at minimal cost, insight about the accuracy of project estimates may be obtained. The impact of that insight may serve to change the capital budgeting plan at some point in the sequential decision making process, as the uncertainty resolution may well cause changes in the economic lives of various projects. Therefore, the problem statement is "Can sequential post-audit information provide uncertainty resolution in cash flow estimation and be utilized in an investment decision making process?"

Research Objectives

The principle objective of this research is to develop the techniques that are needed to obtain uncertainty resolution of initial estimates of cash flow performances for capital budgeting decisions, with specific attention to equipment replacement situations. Because the cash flows may be represented by one of several probability distributions, this research is concerned with techniques for both

discrete and continuous probability functions. Another objective is to develop these uncertainty resolution techniques in a fashion that permits the quantitative incorporation of the strength of beliefs about the quality of the initial estimates and, similarly, the beliefs about the representative quality (or replicability) of the observed cash flow realizations. A final objective is to develop a managerial tool that graphically displays the uncertainty resolution in a timely manner. These objectives are accomplished by:

1. Developing uncertainty resolution techniques that are modifications of existing Bayesian revision models. The Dirichlet distribution revision model is adapted to provide the revision techniques for situations when the cash flow distribution is represented by a discrete probability function. For cash flows that are represented by continuous probability distributions, procedures are developed that modify established beta and normal distribution revision models. These situations take advantage of the natural conjugate properties of the particular distribution's family. When beta and normal distributions do not accurately describe the prior beliefs, or when the sample likelihood and prior beliefs are not natural conjugates, a technique is developed that divides the continuous prior beliefs and sample likelihoods into discrete intervals, and applies the Dirichlet revision techniques.

2. Developing the concept of an equivalent sample size to incorporate the businessman's strength of beliefs about the quality of the sample information, using a multiple of the observed results, to

reflect stronger or weaker beliefs about the representative quality of the sample.

3. Developing the technique of manipulating the distributional descriptive shape parameters (for the Dirichlet and beta revision models) to incorporate the businessman's strength of belief in the quality of his prior estimates, and describe those beliefs. The Information Quality Factor (IQF) is developed for the Dirichlet model, and, in a similar fashion, a technique that uses proportional (α, β) values is developed for the beta distribution model. For the normal distribution, the concept of an equivalent sample size is extended to include the user's prior beliefs.

4. For a management tool, the concepts for Cash Flow Control Charts are developed. These charts graphically portray uncertainty resolution over time. However, the applicability of these charts is limited to situations where there are more than just a few identical projects of interest.

Study Plan

This study was prepared under manuscript format with the principle chapters (III through VI) written as journal articles. Each of these chapters includes a case study of a decision problem handled by an actual manufacturing firm.

Chapter II provides a general review of the literature. Specific issues that pertain to a particular chapter (journal article), are included in the introductory and model development paragraphs of that chapter. The review of the literature reflects the need for uncertainty resolution techniques for the equipment replacement

problem, the need for a means to systematically incorporate the user's strength of beliefs about the quality of his prior and sample information, and the need for a usable management tool.

Because this study is presented in manuscript format, Figure 1 is provided to illustrate how the remaining chapters are related. Capital budgeting methods are the major problem categorization, with equipment replacement decisions as the subset of interest. For this study, these decisions are treated as being made with the existence of both risk and uncertainty. Once implemented, these projects are controlled by methods that either use or do not use post-audit information. Use of post-audit information is the central theme of this paper. The problems handled by these methods have initial estimates that are described by probability functions, either discrete or continuous. The continuous estimates and sample information can be handled with either continuous natural conjugate distributions or by discrete approximations.

Chapter III develops the techniques necessary to obtain uncertainty resolution for replacement cash flows that are modeled as discrete probability functions. The chapter introduces the concept of using the Dirichlet revision model to accomplish this resolution. Additionally, in the development of initial prior beliefs, this chapter shows how the Dirichlet distribution's shape parameters can be used to illustrate the user's strength of belief in the estimates.

Chapter IV develops the techniques necessary to obtain uncertainty resolution for cash flows that are modeled as continuous probability functions with natural conjugate properties existing

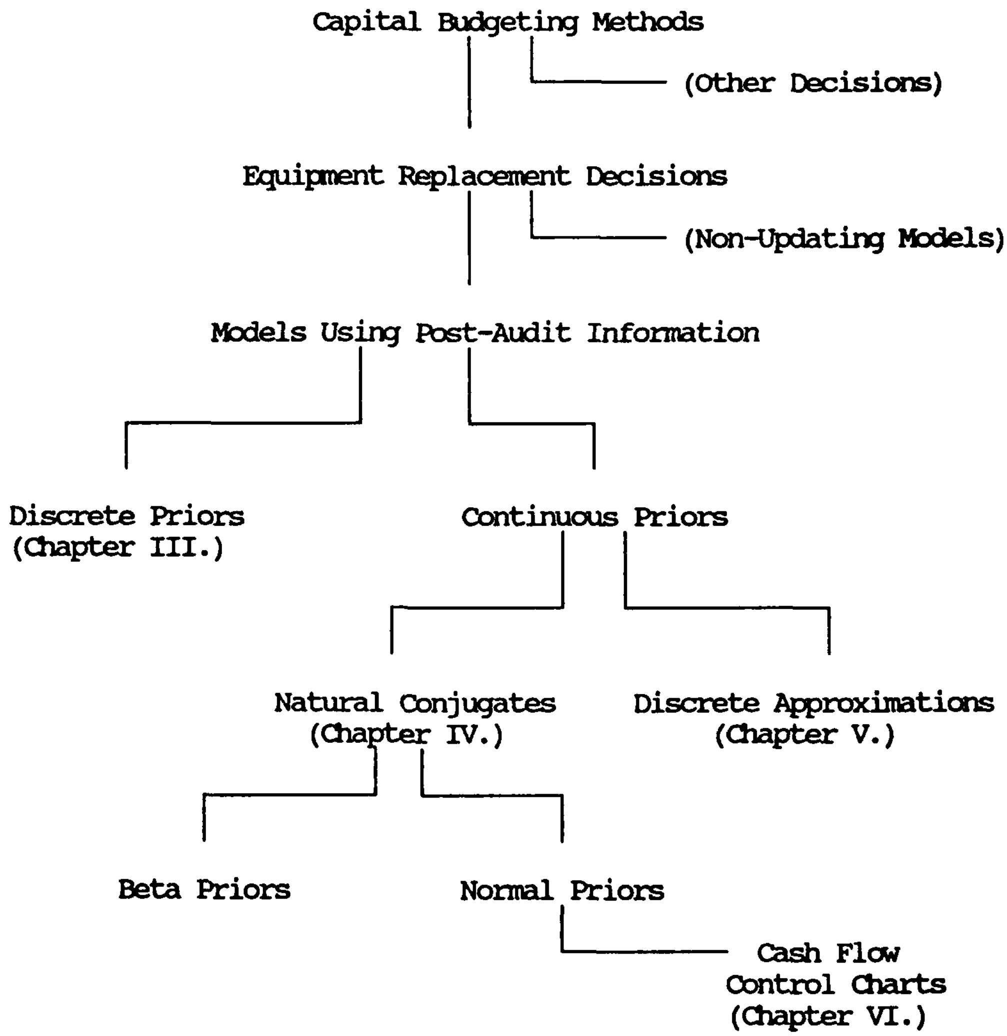

Figure 1. Relationship of Research Topics.

between the sample data and the prior beliefs. In this chapter, a unique method is introduced for the development of the beta distribution shape parameters. This method permits the user to incorporate his strength of beliefs in the prior estimates. The chapter also introduces the concept of an equivalent sample size that can be used to similarly incorporate the strength of prior beliefs for a normal distribution. It then uses this concept to capture the user's beliefs about the quality of his sample information.

Chapter V develops the uncertainty resolution techniques needed when the prior beliefs and/or sample likelihoods do not fit the natural conjugate families described by Chapter IV. This chapter demonstrates how the IQF and equivalent sample size can both be incorporated into a decision problem.

Chapter VI develops the concept of Cash Flow Control Charts as a businessman's management tool for on-going projects. The methods are developed from a blending of statistical quality control charts techniques and natural conjugate distribution concepts.

Chapter VII contains a discussion of the conclusions obtained in chapters III through VI, and recommendations for further research.

II. LITERATURE REVIEW

To be successful in the competitive market, companies must invest their capital in the most advantageous manner possible. Because these companies must contend with varying quantities of unknown future events, they face the generalized problem of capital budgeting under uncertainty. This problem can affect every area of a company's operations, and its need for resolution has led to literary discussions in a variety of disciplines (accounting, business, economics, finance, operations research, and industrial engineering). These discussions have led to the development of numerous capital budgeting models that concentrate on investment selection, but, generally, fail to address the continuous flow of information available from the earnings process.

The generalized problem of an on-going investment with subsequently available cash flow information can be represented as a manufacturing equipment replacement problem. The literature review begins with a general discussion of equipment replacement problems, and how variations of these problems have differing amounts of information about future events. The review then discusses how the amount of available information leads to problem solving by either deterministic (which assume that future events are known with certainty) or nondeterministic (which treat future events as unknown) methods. The review then defines how the terms risk and uncertainty will be used in this study. This permits the nondeterministic methods to be identified

as procedures that either only address risk, or address both risk and uncertainty.

The review then discusses the trend of increasingly available post-audit information. The review categorizes the general capital budgeting nondeterministic methods into models that either use (information dynamic) or do not use (information static) updating techniques. How information static models make adjustments for risk and/or uncertainty is examined, as well as the uncertainty induced bias that results from failing to use post-audit information. For the information dynamic models, how various models incorporate uncertainty resolution are discussed.

The last section of the review focuses on capital budgeting models that are specifically designed for equipment replacement decisions. Historically, the literature has focused on stochastic mathematical programming models, with some applications of computer simulation. The potential for improved company investment performance, by using an application of Bayesian revision for uncertainty resolution is identified and becomes the focus for this research.

Manufacturing Equipment Replacement Problems

As defined in Chapter I, the manufacturing equipment replacement problem focuses on whether a business should keep a defender asset, or replace it with a challenger asset, now or sometime in the future. As further described in Chapter I, this research is particularly concerned with decisions that are made by companies that use periodic decision reviews in their capital budgeting policies. These companies'

investment strategies are affected by the economic lives of their defender assets.

The replacement decisions are based on the estimated cash flows of each alternative. Surveys report that the determination of these estimates is the most difficult of all capital budgeting tasks. However, they also report that equipment replacement decisions usually have the most accurate estimates [Pohlman, Santiago, and Markel (1988)]. These accuracies are associated with the traditional (machine-for-machine) comparison decisions. Currently, as defined earlier, the comparisons have escalated from machine-for-machine to process-for-process decisions. These expansions stem from evolving high technology techniques (such as computers, robotics, artificial intelligence, just-in-time manufacturing, etc.) that incorporate several processes into a single method. The difference between these expanded (process-for-process) comparisons and the more traditional ones is the expanded comparisons do not have the performance "track records" of the traditional comparisons. Surveys report that when a challenger process is a substantial deviation from a firm's previous operations, the previously described cash flow estimation accuracy no longer exists [Cook and Rizzuto (1989)].

The information available under the traditional comparisons led to the development and use of many deterministic methods. The methods proceed as if the decision maker has complete information regarding the investment alternatives, at the time of decision. These deterministic methods dominate the discussions of replacement methods in the literature. The expanded comparisons, on the other hand, do not treat

the estimates as known values, but approach them as random variables with some form of probabilistic future, thus requiring the use of nondeterministic methods.

Replacement Decisions Using Deterministic Methods

Several deterministic capital budgeting models have been developed and are explained in great detail in the literature [Brown and Kritzman (1987), Bierman and Smidt (1980), Weingartner (1963), and Park and Sharp-Bette (1988)]. These discussions center on the model developed by Weingartner that uses mathematical programming (linear, integer, and dynamic) techniques to define the relationships of different variables during the various periods of interest within the planning horizon. The relationships may be simple or complex, and can cover the spectrum of lending and borrowing rate combinations.

One problem associated with these models is that treating all future values as being known with certainty is not always an accurate approach to real-world situations. In fact, it has been stated [Bawa, Brown, and Klien (1979)] that under rapidly changing stock market conditions, these deterministic methods can yield solutions that are sub-optimal moments after they are generated. Another problem is that the accuracy that has historically been attached to the traditional comparisons has led many of the deterministic models to primarily focus on the timing of the replacement instead of possible errors in the estimated cash flows [Bean, Lohmann, and Smith (1985) and Oakford, Lohmann, and Salazar (1984)]. Since most of the expanded replacement decisions will involve cash flow estimates that are stochastic in

nature, the main interest of this research focuses on nondeterministic decision methods.

Replacement Decisions Using Nondeterministic Methods

The predictions of specific future events are made with varying degrees of accuracy, dependent upon the nature of the event and skill of the predictor. The need to reduce the levels of unknown elements in decision making has led to a wide variety of approaches and techniques. To address these concepts, it is necessary to first examine the concepts of risk and uncertainty. For this study, the terms risk and uncertainty must be clearly defined to properly differentiate this research from other previous works, which used these terms interchangeably. Once these terms are defined, the general form of capital budgeting models can be categorized as those that incorporate risk and those that incorporate both risk and uncertainty.

Risk. Risk is defined as the variability between a predicted future event and its actual outcome, where the predictions are probabilistically represented and the distributions have either known, or assumed to be known, parameters. Risk has been historically computed as and represented by the statistical term variance. However, it has been stated [Mao (1970)] that the use of variance as a risk measurement is not entirely satisfactory, as it considers both positive and negative cash flow extremes. The contention is that businessmen are much more sensitive to potential losses than conditions of larger than expected gains. Thereby, that author develops the concept of semivariance, the variability of negative return. It must be noted

that in the situation where an investment is made in combination with other investments, as in a portfolio, the calculation of variance and semivariance requires full knowledge of the covariances between investments (i.e., specific definition of the full covariance matrix, Alexander and Resnick (1985)). However, with many different investments and numerous future cash flows, this computation is difficult, and, as cited previously [Gurnani (1984)], the projects are usually analyzed on an individual basis, with "no allowance made for covariance between projects". Another assessment of risk is made by considering the distribution of net present values, and determining the probability of loss (the area of negative returns). Again, each of these concepts is rooted in knowledge of, or assumed knowledge of, distribution parameters.

Uncertainty. The term uncertainty is used when future events are unknown, and the nature of the probability distribution of event occurrence is also unknown. This area is particularly concerned with judgmental or subjective probability distribution estimates, where the person making the predictions cannot precisely define the distribution parameters. Since most factors can be bounded (either loosely or tightly), the future cash flow estimates are frequently presented as some form of a probability function, with estimated parameters for expected values and variances. These parameters are estimates, and are not known with certainty due to the variability of market or performance conditions.

An immediate extension of an alternative considered under risk and uncertainty is its current and future availability. Some models

assume that investment availability is known with certainty. However, a more realistic approach is that future investment availability is an unknown quantity. Several replacement models have attempted to capture this availability [Chand and Sethi (1982)], but in doing so, they have modeled the cash flows as deterministic values.

Post-Audit Information

As described previously, in realizing a given alternative's true net worth over time, its initial cash flow estimates are replaced by actual cash expenditure and receipt information. This information is readily available within most firms. A review of capital budgeting surveys [Istvan (1961), Bonini (1975), Rosenblatt and Jucker (1979), McInnes, Morris, and Carleton (1982), Gurnani (1984), Klammer and Walker (1984), Mukherjee and Henderson (1987), and Pike (1989)] indicates that an increasing percentage of the firms surveyed have adopted some form of post-audit, to upwards of 90-percent [Klammer and Walker (1984)] of the respondents.

While it is a relatively standard procedure to use this information for accounting reports (stockholders, taxes, etc), some companies use this information as feedback. When used, this feedback is either directed to project originators, the decision makers, or, sometimes, both groups. How previous nondeterministic methods have handled this information can be used to categorize these research efforts. Some models have been developed that do not take advantage of this "free" post-audit information (information static models), while others incorporate this information to varying degrees (information dynamic models). The consequence of ignoring this post-audit

information is that an uncertainty induced bias may be incorporated into the firm's capital budgeting program. The concepts of information static models and uncertainty induced bias will be examined, and then this study will proceed with the case of interest, the information dynamic models and their uncertainty resolution properties.

Information Static Models and Uncertainty Induced Bias

The decision policy of making a single invest/termination decision for a given project typifies the conditions of information static models. These models may address risk only, or both risk and uncertainty. If only risk is considered, then there are several risk adjustment techniques in practice. The most popular techniques used by businessmen are risk adjusted discount rates and adjustment of the required payback period.

Firms currently use two predominant means to determine their required discount, or hurdle rates. The first uses the firm's current weighted average cost-of-capital, and the second uses the marginal cost of new capital. The second method was developed after the correctness of using historical rates, in the face of steadily increasing or decreasing capital costs, was questioned [Bussey (1978)]. Then, simply put, the risk-adjusted discount rate increases the amount of required return on invested capital for investments with greater risk. The drawback to this method is the subjective nature of the adjustment [Park (1977)].

Before considering adjustment of the payback period, its inherent shortfalls should be pointed out. There are three serious

limitations to using payback period as a measure of effectiveness: (1) investment risk is not integrated into the analysis, (2) the timing of net cash flows and their time value of money is disregarded, and (3) post-payback net cash flows are not considered. Reduction of the required payback period creates an additional problem in the development of adjustment consistency from risky investment to risky investment. On the other hand, a more increasingly accepted means of addressing risk is sensitivity analysis, which identifies the factors that generate the risk, and attempts to bound the risk effects.

Since risk has been defined as the variability between predicted and actual events, the most detailed way to represent risk is through the underlying probability distribution of the predicted event (say the net present value distribution). The two approaches that use probability distributions to address this concern are the analytic approach and the simulation approach. The analytic approach [Hillier (1963, 1971)] used the future event probability parameters with a selected interest rate, to determine the exact mean and variance of the discounted value of the investment. The simulation, or Monte Carlo, approach [Hertz (1964, 1968)] uses random sampling from the pre-determined future event probability distributions, discounts them to net present values, and, through repeated simulation, approximates the mean and variance. The complex nature of many real world investment situations often makes the analytic approach difficult, if not impossible, to execute. On the other hand, the inherent flexibility of the simulation approach, combined with more readily accessible computer support, has made simulation increasingly popular. However, both

methods are limited by the required assumption that the probability distribution parameters are known values, thereby ignoring uncertainty.

The recognition that uncertainty exists has led to the realization that a corresponding performance bias exists in static capital budgeting methods. In a previous work [Brown (1974)], it was shown that in the consideration of an unbiased group of investments (i.e., the expected value equals the estimated value for the group of investments), the firm will usually rank the investments on the basis of expected return, and select those with returns above a specified level. In developing the investments' performance estimates, in order for the group to be unbiased, some of the investments with poor performance estimates will have better than anticipated results and some of the higher rated investments will perform worse than expected. Because the non-random nature of the selection process will select the higher ranked investments, the selected group's performance will most likely be less than their initial expected value. Thus, even though the investment estimates were unbiased, an uncertainty induced bias exists.

Another work [Miller (1978)] attempts to overcome this bias through the development of an uncertainty compensating premium (the greater the uncertainty, the greater the required correction). The firm then only accepts those investments whose estimated returns exceeds the required cut-off. However, a key assumption to this approach is that the level of uncertainty is the same for all considered investments (a rare occurrence in real life situations). Lastly, he recommends that these steps be applied only by top level

managers as he feels they would have more information about the population of good and bad investments than the engineer or analyst preparing the investment proposal.

Another primarily information static method that considers both risk and uncertainty is the application of utility theory. The development of utility theory is widely documented in the literature [(Fishburn (1970), Grossman, Kihlstrom, and Mirman (1977), Raiffa (1970), Keeney and Raiffa (1976), and Kreps (1988), for example]. Utility theory addresses the decision maker's (or makers') subjective response to a set of alternatives, given the risks and uncertainties associated with those alternatives. The responses are used to identify the preference or utility, function for that decision maker. The utility function describes the acceptable rate of trade-offs (gains versus variability) for the decision maker and he may be classified as risk averse, risk neutral, or risk prone.

The expected value maximization model [Weingartner (1967)] and the certainty equivalence model [Robichek and Myers (1965) and Percival and Westerfield (1976)] are examples of utility theory applications. However, while this approach is being more widely accepted [Mukherjee and Henderson (1979)], it is difficult to derive the utility function numerically, especially when it is necessary to consider the time value, or preference of the projected benefits (the problem of finding a means to appropriately "discount" the utility values to other points in time). Additionally, it is generally even more difficult to explain those functions to decision makers that do not possess utility theory

backgrounds, and those individuals usually prefer to use monetary indices.

Each of the methods mentioned can only attempt to compensate for the factor uncertainties, because they have no means to clarify them. While using some form of uncertainty compensation premiums may be a way to resolve the problem in information static models, feedback (post-audit) information can provide a more efficient means to overcome the uncertainty and makes the decision a dynamic, rather than static process.

Information Dynamic Models and Uncertainty Resolution

The concept of uncertainty resolution, as defined in chapter I, has not received much attention in the literature as an application to capital budgeting models, in general, and even less for equipment replacement problems, specifically. With regards to the general models, the previous works have primarily sought to develop indices to measure the attainment of uncertainty resolution. These efforts have examined using the payback period, the coefficient of variation, certainty-equivalence, and the project balance concept.

The use of the payback period was proposed, based on a simple, one decision investment, with the amount and timing of the cash flows modeled as random variables [Weingartner (1969)]. This work defined the time required for expected cash inflows to equal the initial capital investment to be the rate of expected uncertainty resolution (which is also the definition of the payback period). However, this work failed to elaborate on how this concept could be extended to

multi-period decision problems. It also failed to address the inherent weaknesses in using the payback period: (1) the time value of money is ignored, and (2) post-payback net cash flows are ignored.

When the coefficient of variation was introduced as a measure of uncertainty resolution [Van Horne (1969)], it was proposed that the square root of the weighted average (based on probability of occurrence) of the variance (about the conditional mean) of probability tree outcomes, at the end of any time period of concern, be divided by the expected net present value of the cash flow, thus forming a coefficient of variation ratio for each period of time. This was extended to the portfolio situation, where the coefficient of variation was used as a gage to the investor who desires to maintain a particular risk profile. This method required developing the coefficient of variation patterns, over time, for portfolios with existing products and alternative portfolios that included new products. A corresponding coefficient of variation differential chart is then developed, and, based on the anticipated differential, it is determined when the portfolio uncertainty is resolved enough to permit new investments.

In a critique of this work [Bierman and Hausman (1972)], counter-examples are given that show that the coefficient of variation technique does not provide complete information. The critique goes on to say that entire probability distribution (and not just mean and variance) is needed to provide full information for the investment. In the discussion of the portfolio approach, the critique agrees that under the condition where the occurrence, timing and characteristics of future investments are not known with certainty, Van Horne's

uncertainty resolution may have merit in maintaining a given risk profile. However, they contend that if the purpose is to maintain a risk posture, an investor would be indifferent between a set of investments that never resolve their uncertainty and a program that acquires "new" investments each year that fully resolve their uncertainty each year.

Another effort [Park (1977)], elaborates further on the weaknesses of the coefficient of variation technique. It examines the measure's dependence on the terminal net present value, and its subsequent failure to consider the cash flow pattern of the probability tree. The effort provides an example of two probability trees, each with identical terminal values and conditional probabilities. By Van Horne's technique, this situation will result in equivalent resolution of uncertainty over time. However, the construct of the two alternatives is such that one recovers its initial investment faster than the other. Thus, under conditions of uncertainty, the shape of the cash flow pattern is important because it provides information about the rate at which outcome uncertainty is resolved.

The project balance measure was introduced [Park (1977)] as a measure of investment worth and uncertainty resolution. The measure is a time sensitive application of incoming cash flows to an investment's unrecovered balance of capital. Simply stated, an investment's unrecovered capital balance is carried forward through time, being steadily incremented at the firm's minimum attractive rate of return, and decremented by the arrival of cash inflows. The process continues after the initial investment is recovered, running throughout the

course of investment life. In graphic portrayal, this develops areas of negative balance (ANB), corresponding to times when the cumulative incoming cash flows do not exceed the amounts of investment, and areas of positive balance (APB), corresponding to times when the cumulative incoming cash flows exceed the amounts of investment. The project balance concept simultaneously captures the payback period information and the future worth information (another form of discounted cash flows).

The author then turns to the matter of uncertainty resolution, and, considering the shortcomings found in Van Horne's approach, applies the project balance concept to the coefficient of variation. To overcome the limitation created by Van Horne's use of terminal project value, Park substituted the time sensitive measures ANB and APB. When these measures were applied to Bierman and Hausman's counter-example (two investments yielding identical mean and variances but one having a shorter payback period), the results showed that Park's resolution indices could be more discriminating than Van Horne's (which could not discern any differences). In further explanation, the author addresses the problem created whenever any standard deviation based measure is used. Specifically, as a measure of variability, the standard deviation treats any fluctuations from the expected value as bad, but for risk-averters, fluctuations above the expected value are not nearly as bad as those below. He overcomes this problem through an enhancement based on Baumol's expected gain confidence limit [Baumol (1963)]. The result is a time-dependent measure of uncertainty

resolution, and a subsequent application was provided [Park and Thuesen (1979)].

An alternative approach is to attain uncertainty resolution through probability distribution revision by Bayesian techniques. These techniques are well documented in the literature [Chernoff (1968), Chernoff and Petkau (1986), Hey (1983), Iversen (1984), Lindley (1965), West (1986), Winkler (1972), Zellner (1985), for example]. Earlier works [Magee (Jul-Aug 1964, Sep-Oct 1964)] laid the groundwork for Bayesian decision tree investment analysis. The basic procedural steps were to: (1) identify the problem and alternatives, (2) layout the decision tree, (3) obtain likelihood data, and (4) evaluate alternative courses of action. These works were limited to discrete event situations, addressing continuous alternatives by selectively breaking them into intervals. Another work [Hespos and Strassman (1965)] addressed this problem of continuity through their development of the stochastic decision tree, which used simulation to derive the final distributions. However, these initial efforts had an inherent limitation, as they treated the parameter estimates as known values. This effectively addressed the risk elements, but failed to address questions concerning uncertainty.

Subsequently, it was proposed that updating information should be periodically incorporated into a given project's valuation considerations, because the capital budgeting process is sequential in nature [Harpaz and Thomadakis (1984)]. In this method, an updated valuation formula is found via Bayesian methods, and it is contrasted

with conventional results in a comparison of two projects over two time periods.

Another effort developed the capital growth potential criterion for a general multiperiod capital budgeting model [Park (1987)]. Under this concept, an investor selects projects, uses Bayesian revision to resolve the cash flow uncertainties quickly, and, thereby, gains greater capital flexibility for reinvestment opportunities. This effort combined Bayes theorem with binomial sampling to attain uncertainty resolution. This effort found that post-audit information should be incorporated into evaluations for project investment.

Other works with capital budgeting models, in general, have shown that these methods can be particularly beneficial when the updating information can be used with replicable investments [Bierman and Rao (1978) and Cyert and DeGroot (1987)]. The facts that many equipment proposals include performance estimates, that many companies employ periodic decision review policies, and that post-audit information is available make these decisions fit the traditional Bayesian decision framework.

Other applications of these techniques cover many areas, including auditing applications [Crosby (1985) and Lin (1984)], depletive and non-depletive inventory models [Azoury (1985)], competitive bidding models [Attanasi and Johnson (1975)], price expectation models [Turnovsky (1969)], work sampling [Buck, Askin, and Tanchoco (1983)], quality control [Hoadley (1981)], as well as many other works that handle some form of directly measured data.

Techniques for Solving Equipment Replacement Problems

A principle goal of this research is to develop a methodology that considers and resolves the uncertainties involved in an equipment replacement problem where the firm uses a periodic decision review policy. While it is apparent that Bayesian methods have great potential benefit, these techniques are not presently being used for replacement decisions. These decisions are primarily being handled by two analytic techniques, stochastic mathematical programming and computer simulation.

Stochastic Mathematical Programming Methods

The stochastic programming models are based on Weingartner's horizon model, and expand it to address the variables' probabilistic risk. An earlier work [Lockett and Gear (1975)] developed an integer programming model that used stochastic decision trees to represent sequences of events. They also state that this model may require either simulation formulation or relaxation of the integer programming approach to a linear programming model as the number of integer variables increases. Another approach [Salazar and Sen (1968)] introduced a stochastic linear programming model that considered risk by repeated simulation of future events, after random selection of a subset of investments, or trial portfolio, from the investment population available. This results in a value distribution for each trial portfolio, and this is used to generate a risk-return chart. If the distribution of returns is not high enough, that trial portfolio is unacceptable. The problem with both of these models is that the

detailed information required to construct factor constraint relationships may be difficult to accurately determine in real world situations. Further, each assumes that the appropriate probability distributions are known. (Although the Salazar and Sen work was entitled "A Simulation Model of Capital Budgeting Under Uncertainty", they only address risk in their model.)

Another model [Prastacos (1983)] attempted to combine dynamic programming with probability distribution considerations, but it only considered the most likely value of the probability function in its formulation. As a result, this method does not capture the information provided by the pattern of the probability distributions. In an extension of that method, a model was introduced that combines dynamic programming with computer simulation [Lohmann (1986)]. The model uses the equipment replacement problem with the cash flows and availability of alternatives modeled as triangular probability distributions. The dynamic programming formulation is repeated solved, based on samples drawn from the simulated distributions, and the resulting analysis seeks the optimal replacement sequence and its timing. This model has flexibility, but its formulation considers the estimated triangular distributions to be the true distributions of events, and makes no adjustments to these distributions over time. Thereby, this model addresses risk, but ignores uncertainty.

Computer Simulation Methods

The use of computer simulation has become more popular as alternative investments and company investment situations have become increasingly complex. Computer simulation permits many factors to be

included in the problem formulation. However, these applications have predominately focused on risk considerations, and have no means to explicitly incorporate uncertainty and post-audit information.

The first attempted use of simulation models as an investment analysis tool [Hertz (1964, 1968)], followed the fundamental steps: (1) key factors (to include investment alternatives) be identified, (2) probabilistic natures, or risk profiles be developed for these factors, (3) randomly select a subset of factors, (4) simulate performance (using a 15-year horizon), (5) compute yearly results, (6) replicate until stable, and (7) repeat for all policy subsets. The short fall in this approach is that it considers the estimates of the probabilistic parameters to be the known values.

A previous work attempts to address uncertainty through an application of sensitivity analysis [Lothner, Hoganson, and Rubin (1986)]. The authors recognized that the parameters were just estimates, and correspondingly developed optimistic and pessimistic sets of parameters to go along with the most likely set. The simulations were replicated for all three sets, and the three resultant outcome distributions were statistically compared. A similar approach was used in a portfolio management simulation model [Bradley and Crane (1975)].

Bayesian Methods

An exploratory work [Snyder (1988)] examined the potential gains that uncertainty resolution, via Bayesian methods, could provide to a specific equipment replacement problem. This effort compared the cash flows of a non-updating replacement sequence with a sequence that used

post-audit information. The effort considered a situation where the cash flows were modeled as normal probability distributions (in a situation where the sample variance was assumed to be the true population variance). This effort also examined a specific investment with sampling problem.

The importance of post-audit information and uncertainty resolution to capital budgeting models, in general, has been addressed by several authors. However, with the exceptions noted above, nowhere in the literature is there a detailed application of that concept to equipment replacement projects when periodic decision reviews are made for those current projects. Therefore, the purpose of this research is to investigate this situation.

III. INCREMENTAL AUTOMATION WITH SAMPLING - APPLIED TO AN ADVANCED MANUFACTURING SYSTEM

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ABSTRACT

Capital budgeting models for project selection have been extensively explored, but relatively little attention has been given to the aspects of post-audit and project control for the projects implemented. This paper examines the area of project control and proposes a method that incorporates post-audit information as an active element in the decision to maintain or terminate an initiated project. Project selection is primarily based on anticipated performance, as determined from the most accurate information available. However, in a sequential decision environment, information that is accurate at one point in time may become inaccurate at another. This paper will show how the Dirichlet distribution can be utilized to formulate appropriately weighted prior probability beliefs, and how these initial beliefs can be updated as we receive post-audit information. To do this, we incorporate categorized cash flow data in a unique Bayesian-based framework. To illustrate the use of the Dirichlet distribution, we present a case study of an actual automation decision for a flexible manufacturing system. This case study also demonstrates how decision strategies can be improved by using post-audit information, when compared with conventional methods.

INTRODUCTION

High equipment costs, particularly for high technology computer and robotics equipment, combined with unknown future performances and capital budget limitations have made manufacturing investment decisions increasingly difficult. These investment decisions include both project selection (the evaluation and comparison of alternatives) and project implementation and control (the identification of deviations from projections and subsequent evaluation and correction). While capital budgeting models for project selection have been extensively explored, relatively little attention has been given to the aspects of project post-audit and control.

Investment decisions require the most accurate information available, but what may have been accurate at the initial selection time is not necessarily accurate for subsequent decision points. The information collected during the passage of time from initial selection to a subsequent decision point updates the previous level of information. This paper presents a systematic updating method that takes advantage of the flexibility characteristics of the Dirichlet distribution, in a unique application to a sequential automation problem for a typical manufacturing system. This distribution, when used with a multinomial likelihood function, provides a natural basis for solving a decision problem with categorized outcomes (excellent,

fair, poor, etc.). An understanding of the following general concepts will provide some insight to the discussions to follow.

In many nondeterministic capital budgeting models, the terms risk and uncertainty are often used interchangeably in the discussion of unknown future cash flows estimates [Magee (Jul-Aug 1964, and Sep-Oct 1964), Hespos and Strassman (1965), and Salazar and Sen (1968)]. However, in this paper, we treat these terms as distinct expressions. We define risk as the variability between a predicted future event and its actual outcome. These predictions are probabilistically represented, and the distributions have known, or assumed known, parameters. Risk has been historically computed and represented by the statistical term variance. Uncertainty describes future events that cannot be predicted with certainty, and the exact nature, or the descriptive parameters, of their occurrence probability distributions is not known. Many analytic techniques will treat the distribution parameters as known values in their risk assessments, thereby ignoring the uncertainty in the unknown probability distribution. The proper way to address prediction errors is through the process of uncertainty resolution, which is defined as the process of moving from greater uncertainty toward less uncertainty [Bierman and Hausman (1972)]. As a project moves from its inception through its economic life, the original cash flow estimates are replaced by actual cash flow realizations, and the prediction uncertainty is reduced as the project's real performance distribution comes into sharper focus.

Uncertainty resolution is only possible if cash flow post-audit information is available. A review of capital budgeting surveys

indicates that an increasing percentage of firms have adopted some form of post-audit [Istvan (1961), Bonini (1975), Rosenblatt and Jucker (1979), McInnes, Morris, and Carlton (1982), Gurnani (1984), Klammer and Walker (1984), Mukherjee and Henderson (1987), and Pohlman, Santiago, and Markel (1988)]. This post-audit information consists of reports to the management staff concerning expense and receipt realizations that are experienced by the project during its execution. This project data could be used to create an information dynamic model that is based on Bayesian techniques. These techniques are well documented in the literature [Lindley (1965) and Winkler (1972), for example].

Uncertainty resolution, via Bayesian revision, is most explicit when applied to a situation involving project repeatability. Multi-plant firms or single plants with multiple production lines that must implement organization-wide projects can use an incremental approach to their investments [Bierman and Rao (1978)]. The firms can use a portion of the organization as a test vehicle for the project, and use the observed success or failure to update their original proposal estimates. Uncertainty resolution also occurs in "all or do-nothing" scenarios, however, these situations do not have periodic decision points where the firm can react to the revised information.

Another concept that has great impact on this Bayesian method is project abandonment [Kee and Feltus (1982)]. To increase its investment flexibility, a firm must consider its available capital budget (cash held as equity and any amounts borrowed) and its marketable real assets (properties and currently held equipment) as

company capital resources. The liquidation of a currently held asset requires a decision to abandon that asset in its current capacity. The abandonment decision terminates a particular project or investment, and it has an associated abandonment payoff that is the market salvage value of the project or investment minus its associated expected return, discounted to a specific point in time. Project abandonment will require varying degrees of execution effort, dependent upon the specific manufacturing situation. If the situation requires retention of overall production capacity, the abandonment process must include the acquisition of an alternative, capacity-maintaining piece of equipment. The abandonment payoff, in this complex situation, will include the salvage value of the old equipment, new equipment costs, and expected return from the new equipment. Simpler abandonment situations may involve either the retiring of a given piece of equipment or the termination of a multi-phase project at some intermediate point, resulting in multiple production processes. Exactly how abandonment is handled requires many managerial inputs, and an abandonment decision that results in mixed methods of production will have more than just financial considerations. Product quality must be addressed. Should the firm keep the converted equipment or bring back the old equipment, as well as various other questions. In fact, the situation of mixed production modes may be unacceptable for many firms, but those aspects are not intended for discussion in this paper. The key factor in any abandonment decision is that the net gain from the abandonment payoff is additional capital for reallocation.

DECISION MODEL

In this section, we will present a general decision framework for incremental automation with sampling.

Assumptions. Our method assumes that the following conditions exist during a given project's life:

(a) The firm makes regular periodic reviews of its investments.

(b) The firm makes sequential project implementations, when the nature of the project indicates that it is appropriate.

(c) Cash flow estimates are represented as probability distributions, either discrete or continuous (realizing that the distribution parameters are not known with certainty).

(d) Post-audit information on project performance is available and immediately reported to management for the periodic reviews. This information is the key to this revision process, as the variations of this data will be the key factor in the determination of whether or not to revise the initial estimates.

(e) Future investment opportunity availability is not known with certainty. The alternatives that are available for consideration at a particular point in time may not be the same at some other point.

Theoretical Model. The modeling concept used in this paper is to apply distribution parameter revision to incremental automation for a typical manufacturing process. The initial adopt or delay decision for automation is based on the best information available at time 0, and is

determined through conventional procedures. However, at the subsequent review and decision points, the impacts of parameter revision can be realized. The problem is formulated, with particular attention being given to:

(a) Cash flow distribution estimations: the cash flow estimations are represented by probability distributions with uncertain parameters. As newer information becomes available, these parameter estimates will most likely change during the project's economic life.

(b) Quality of the sample post-audit information: determinations must be made to assess the predictive quality of the sample information and how well the firm can interpret that information. The assessment of its predictive value will include an examination of the project's market environment to determine the existence of any anomalous incidents or trends. The assessment of interpretation accuracy will examine the likelihood of analytic reports, given the existence of particular true conditions. The purpose of these assessments is to prevent inappropriate (or premature) reactions to the sample data.

(c) Probability distribution revision process: the updating process will use the Dirichlet-multinomial family of natural conjugate distributions. The Dirichlet distribution is sometimes referred to as the multivariate beta distribution, and this method can be thought of as a generalization of the beta-binomial conjugate family. In the broadest terms, the process involves the formulation of a Dirichlet prior distribution, sampling to develop a multinomial likelihood

function, with a resultant determination of a Dirichlet posterior distribution.

In the problem formulation, it is given that any particular result will fit into exactly one of k different outcome categories. The occurrence probabilities associated with each of the k respective categories comprises the random vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ with $(\theta_j \geq 0;$ $j=1, \ldots, k)$ and $\Sigma\theta_j = 1$. There also exists a parametric shape vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ with $(\alpha_j > 0;$ $j=1, \ldots, k)$, such that the probability density function of $f(\boldsymbol{\theta} \mid \boldsymbol{\alpha})$ will form a Dirichlet prior distribution

$$f(\boldsymbol{\theta} \mid \boldsymbol{\alpha}) = \frac{\Gamma(\alpha_1 + \ldots + \alpha_k)}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_k)} \theta_1^{\alpha_1 - 1} \cdots \theta_k^{\alpha_k - 1} \qquad (1)$$

where,

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt$$

and the α_j are not limited to integer values, and if k=2, the result is a beta distribution. Then, the observed sample is a random vector $\mathbf{x} =$ $(x_1, \ldots, x_k)$ and $(x_j \geq 0; j=1, \ldots, k)$, where a given x_j represents the number of observations falling into a kth category, with $\Sigma x_j = n$. This observed sample uses the vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$, to form a multinomial likelihood function

$$f(\mathbf{x} \mid n, \boldsymbol{\theta}) = \frac{n!}{x_1! \ldots x_k!} \theta_1^{x_1} \cdots \theta_k^{x_k} \qquad (2)$$

This likelihood function combines with the prior distribution to form a Dirichlet posterior distribution

$$f(\theta \mid \alpha, x) = \frac{\Gamma(n + \Sigma\alpha_i)}{\Gamma(\alpha_1 + x_1)\ldots\Gamma(\alpha_k + x_k)} \theta_1^{\alpha_1+x_1-1} \cdots \theta_k^{\alpha_k+x_k-1} \quad (3)$$

This permits direct computation of the prior and posterior values of θ (θ_j', and θ_j'', respectively) from the Dirichlet distributions' shape parameters [Johnson (1960), and Johnson and Kotz (1969)]:

$$\theta_j' = \alpha_j / \Sigma\alpha_j \quad (4)$$

$$\theta_j'' = (\alpha_j + x_j) / (n + \Sigma\alpha_j) \quad (5)$$

Before going further, there are two items that deserve attention.

The first item, and the most important underlying concept in using the Dirichlet distribution, concerns the development of the distribution's initial descriptive parameters. The natures of the α_j's are such that the stronger one believes a parameter is true, the larger its corresponding α_j. This belief may be subjective or objective in origin. For example, a given Dirichlet distribution has three equiprobable states of nature ($\theta_1 = \theta_2 = \theta_3 = 1/3$). These probabilities can be described by $\alpha = (100,100,100)$, a strongly held prior belief, or by $\alpha = (1,1,1)$, a weakly held prior belief. Ten units are sampled, with all ten outcomes favoring θ_1 ($n = 10$, $x_1 = 10$, $x_2 = x_3 = 0$). For the parameter set, $\alpha = (100,100,100)$, the posterior values for θ_j become

$$\theta_1'' = \frac{\alpha_1 + x_1}{n + \Sigma\alpha_j} = \frac{100 + 10}{10 + 300} \approx 0.355$$

and,

$$\theta_2'' = \theta_3'' = (100 + 0)/(10 + 300) \approx 0.323.$$

On the other hand, using the second set, $\alpha = (1,1,1)$, yields

$$\theta_1'' = \frac{1 + 10}{10 + 3} \approx 0.846$$

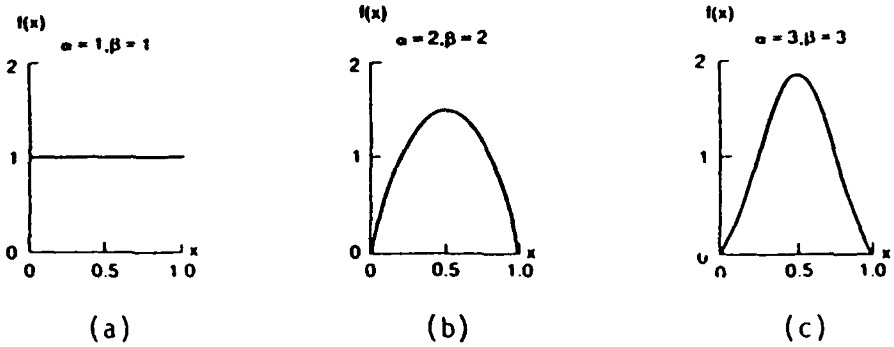
and,

$$\theta_2'' = \theta_3'' = (1 + 0)/(10 + 3) \approx 0.077$$

The second shape parameter set reacts very quickly to the sample observations, while the first set is hardly disturbed. Perhaps an easier way to visualize the problem is to consider a case where $k = 2$, the Dirichlet distribution is now the more familiar beta distribution. Figure 1 shows three beta distributions, each with an expected value of 0.5 ($E(X) = \alpha/(\alpha + \beta)$, for beta distributions), but it is readily apparent that part (c) is more weighted, or is "tighter", to the 0.5 value than, say, part (a) or (b).

The estimation of the Dirichlet shape parameters raises two important questions: (1) "What are the predicted outcome probabilities for the various states of nature?", and (2) "How strongly do we believe the predictions will hold true?" These are difficult questions, and it has been noted that the relative proportion elicitation becomes increasingly difficult for the decision makers, when the states of nature exceed four [Bessler and Chamberlain (1988)]. Proper handling of the relative weights requires a thorough understanding of the Dirichlet distribution properties by staff and management personnel.

The second item focuses on earlier works with this distribution. Some of the developmental works with this natural conjugate distribution family were focused on goodness-of-fit and independence tests [Good (1967)]. These works used symmetric (equiprobable) Dirichlet distributions, with the notation



$$f(x;\alpha,\beta)=\begin{cases}\dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, & 0<x<1;\alpha>0;\beta>0\\[2ex] 0, & \text{otherwise}\end{cases}$$

$$E(X)=\frac{\alpha}{\alpha+\beta}$$

Figure 1. Various beta distributions with a common expected value (0.5), but having markedly different shapes, due to different descriptive shape parameters (α, β).

$$f(\theta \mid \alpha) = \frac{\Gamma(\alpha k)}{\Gamma(\alpha)^k} \theta_1^{\alpha-1} \ldots \theta_k^{\alpha-1} \tag{6}$$

where,

$$\alpha_1 = \alpha_2 = \ldots = \alpha_k = \alpha$$

and the α was referred to as a "flattening" constant. When this equiprobable condition was used in a situation where there was a natural grouping of categories (grouping a continuous variable), it was shown that regeneration would eventually lead to loss of the Dirichlet properties [Good (1965)]. However, this problem was overcome by adding a distribution for α (maintaining the α_i's as distinct elements) which leads the posterior distribution away from equiprobability [Lindley (1980), and Good (1983)]. There were no such problems when an asymmetric prior distribution was used, nor were any problems noted when working in a noncontingency table structure (a three category example, using "Prefer A", "Prefer B", "no preference", is discussed in Draper, Hunter, and Tierney (1969)). In our presentation, we will employ the generalized shape parameter $\alpha = (\alpha_1, \ldots, \alpha_k)$ to avoid potential problems, particularly since equiprobable event outcomes are not generally anticipated as final distributions.

(d) Identification of decision strategies: at every periodic review, the company must fully identify its set of investment alternatives. This includes the retention or termination of any current projects, as well as the selection of any new projects.

(e) Areas for sensitivity analysis:

(1) Project abandonment values: these values are estimated for each decision point over the planning horizon, subject to periodic revision, as necessary.

(2) Selection of a discount rate: a particular value (the firm's minimum acceptable rate of return, MARR) is used in the initial analysis, but its effect on solution robustness should be evaluated.

(3) Selection of project planning horizon: this is based on the firm's requirements, and the economic life of each alternative.

Model Synopsis. Once formulated, the adopt/delay decision problem is solved for an initial strategy. Standard project evaluation techniques (net present value method) are applied to reach this initial solution. The importance of our proposed technique begins once the initial strategy has been selected, and the project is implemented. Cash inflows (receipts) and outflows (expenses) occur, and this data is recorded (usually by an accounting department) for tax and stockholder reports. Because the firm must collect this data for other purposes, it can be viewed as essentially "free" information. This information, and its use in this method, does not represent an additional cost, unlike the costs associated with information collected from product testing. (While the collection is "free", there may be some compilation costs if, as we propose, the information is used for more than their original purposes.) The collected data can be used to revise the original distribution parameter estimates through the use of the Dirichlet-multinomial model. This revision process must be accomplished prior to reaching the next decision point, where the problem is readdressed, based on the revised cash flows and this new

point in time. The uncertainty resolution provided by this process may change the initial decision strategy, thereby affecting the firm's overall capital budgeting plan.

INCREMENTAL AUTOMATION: A CASE STUDY

The following case study, using data from an actual manufacturing firm, is provided as an illustration of the Dirichlet-multinomial model. The model requires sample information before it can be utilized. However, rather than skip directly to that point in the case study, there are some supporting points that deserve attention. Incremental, or sequential analysis, along with Bayesian techniques have not been widely used by firms considering the adoption of new manufacturing technologies. Therefore, we have included those details in the case study presentation, to emphasize their contributions.

Decision Problem. The Linkup Corporation is a large manufacturer of shaft and pipe couplings. Their products, even though each type of coupling is produced in a variety of sizes, can be described in three basic categories: (1) gear-type couplings, with sleeves and rigid hubs, (2) grid and flexible hub couplings, for smaller shafts that are laterally or angularly misaligned, and (3) larger F-style and T-style flexible hub couplings. They have two nearly identical factories, one in Atlanta and one in Milwaukee. The factory layout designs were based on anticipated large economic order quantities (EOQ), with equipment that required long set-up and changeover times. However, order quantities were smaller and more frequent than anticipated, and this caused the corporation to carry large amounts of the various types of

inventories (raw materials, work-in-progress (WIP), and finished goods) to meet their customers' demands. The corporation is considering changes to its operating procedures. They want to compare the economic efficiency of their present machine shop operations, see Figure 2, to a proposed flexible cellular manufacturing concept that will use robotics (to reduce labor costs), flexible manufacturing processes (to reduce set-up and changeover downtime), and just-in-time manufacturing (to reduce inventory holding costs). The proposal indicates that each factory can be converted into three modules (six total), with five robotic cells in each module. Figure 3 is a schematic diagram of the proposed conversion of both factories, and Figure 4 shows the proposed layout for one five-celled module. The differences from cell-to-cell are minimal, consisting mostly of different tooling sets. The proposed layout is structured to have each module produce all three families of product. This layout was designed with a phased, or incremental, conversion in mind. Under those conditions, the factory's entire output of one product family is not dependent upon the new technology, thus reducing potential losses. The Engineering Department input predicts cellular conversion results as a discrete set of three outcomes:

Conversion result (notation)		Probability of outcome
Excellent	(θ_1)	$P(\theta_1) = 0.4$
Fair	(θ_2)	$P(\theta_2) = 0.4$
Poor	(θ_3)	$P(\theta_3) = 0.2$

At this time, it is felt that while there is tremendous success potential for this flexible manufacturing system, there is no historical data available as input to the probability determination.

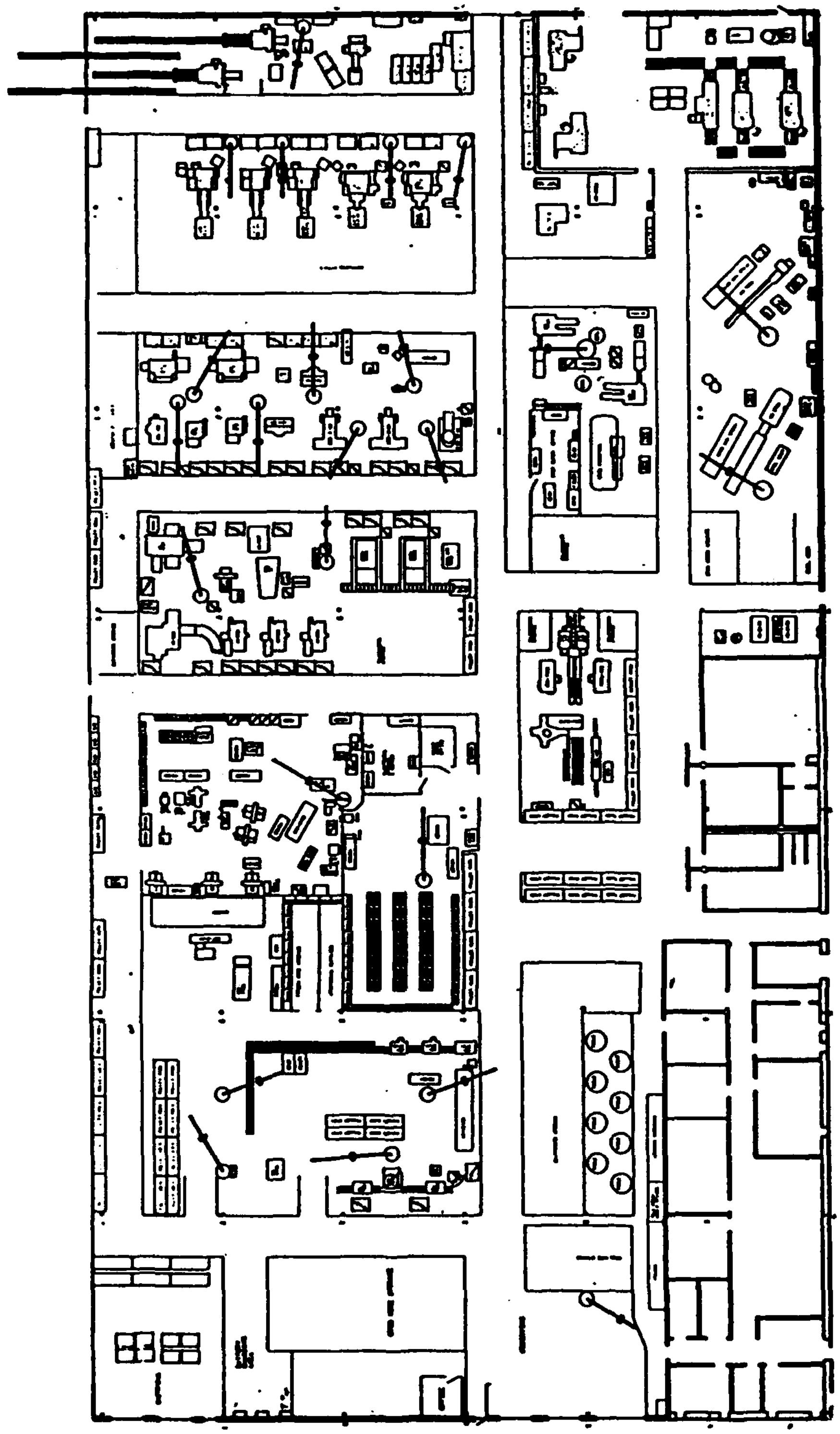


Figure 2. Current machine shop layout in each factory.

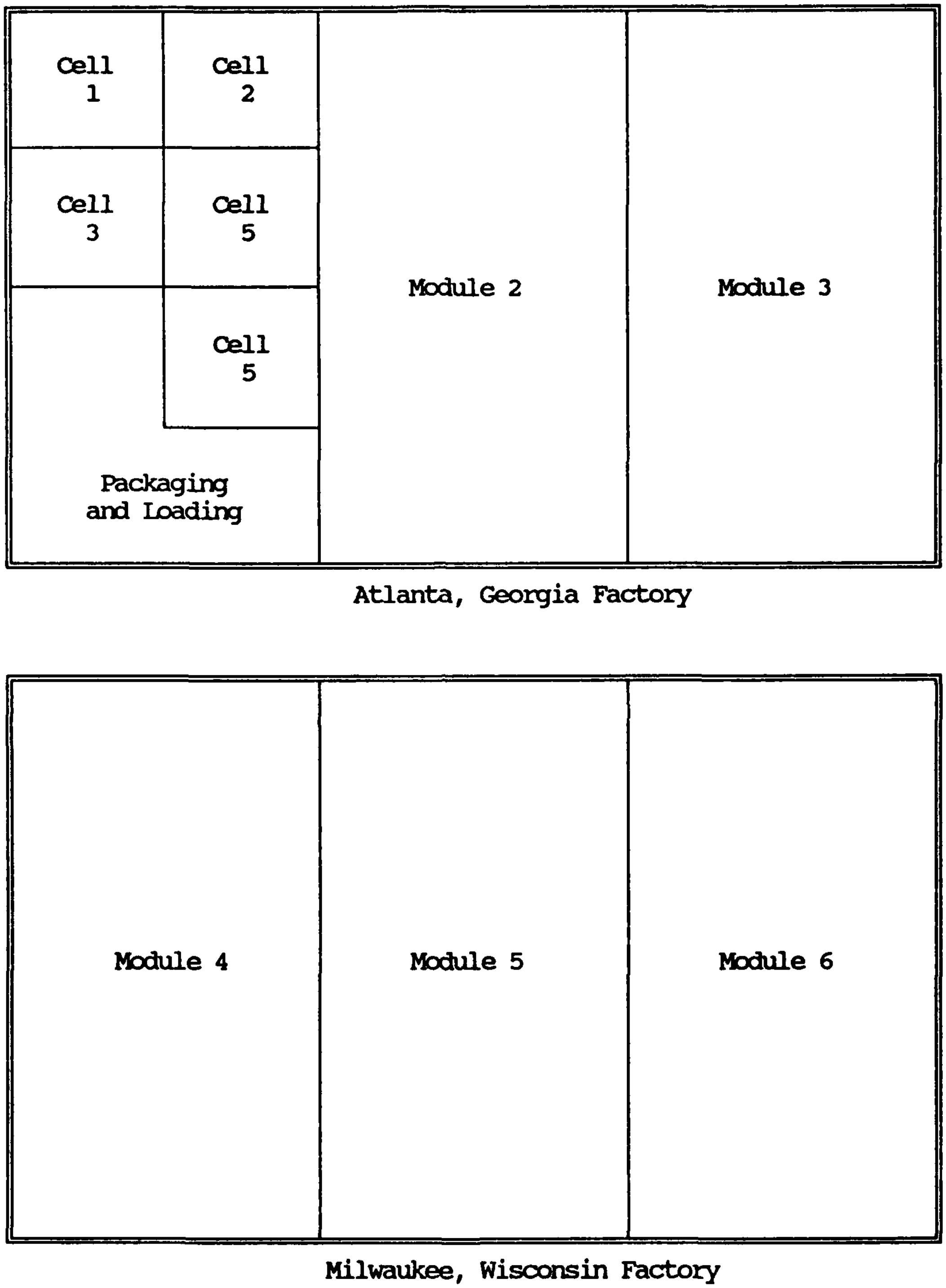


Figure 3. Schematic diagram of the Linkup Corporation's production assets, under the cellular manufacturing concept.

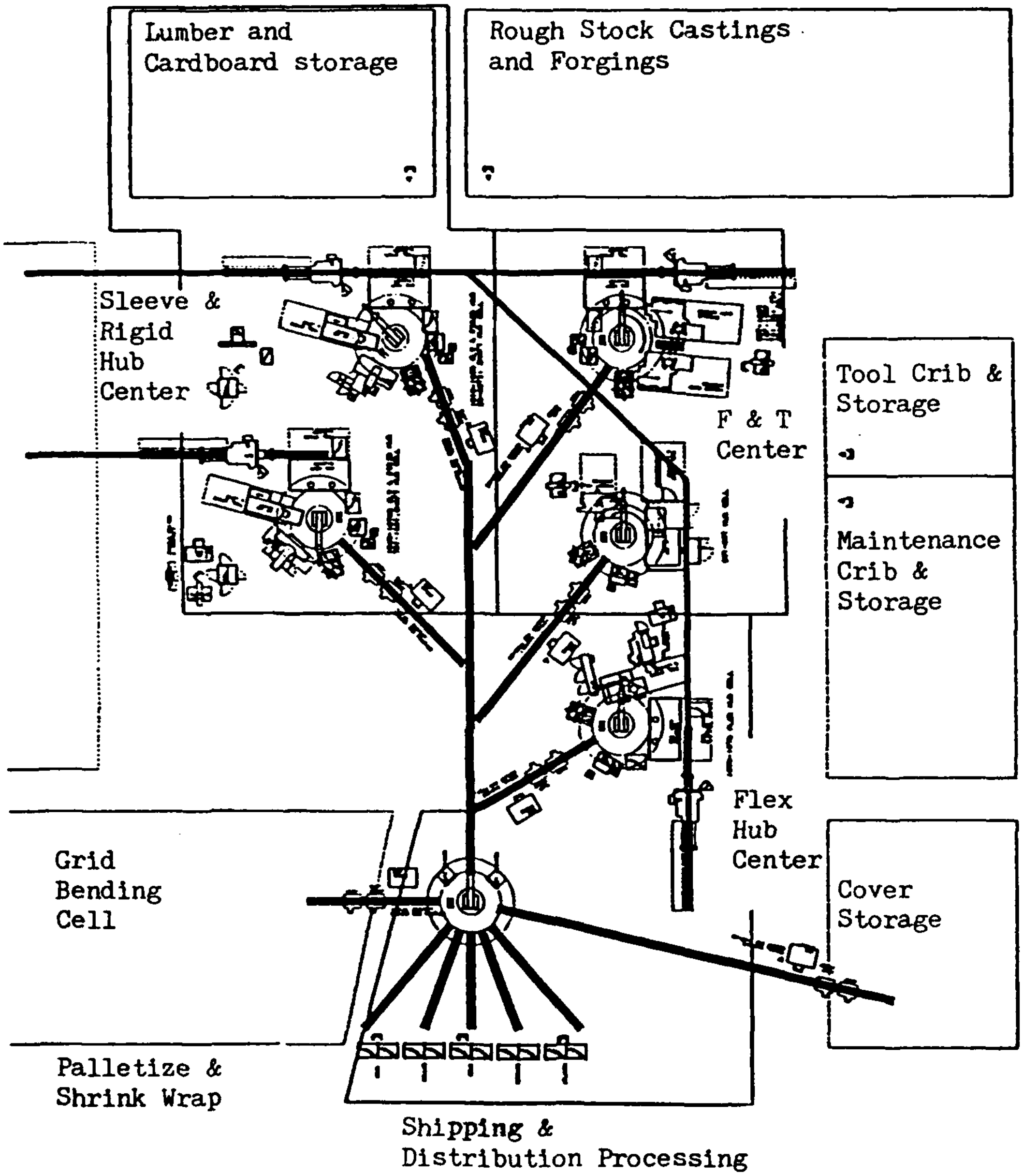


Figure 4. Proposed layout for one five-celled module.

However, it is anticipated that the changeover should be very straightforward and sample results for any one cell could be representative of any other cell. Therefore, the descriptive parameters for the Dirichlet prior distribution were set as $\alpha = (1., 1., 0.5)$ (note $\Sigma\alpha_i = 2.5$ and $1./2.5 = 0.4$ and $0.5/2.5 = 0.2$).

The equipment needed for conversion will be available for purchase during the next two years. The investment cost for one module is presently $189,200. However, with improving computer production technology, the actual cost for the computer components associated with this equipment is expected to be reduced by $5,000 per year, during the next two years.

The forecast for product demand leads the firm to adopt a 10-year planning horizon for this project, and their minimum acceptable rate of return (MARR), or cost of capital, is 10-percent, after taxes. The projected after-tax cash flows for a converted module, and their respective discounted values, are listed in Table 1. In preparing these cash flows, the Marketing Department assumed that all cells in a given module would have the same conversion result, and that each module would handle one-third of the factory's projected requirements. All equipment salvage values were treated as negligible after eight production years.

The two-year equipment conversion window allows the corporation to consider incremental conversion alternatives, along with the more traditional "all or do-nothing" alternatives. The incremental alternatives have the characteristics of sequential decision making processes: a module is converted, its results interpreted, and

Table 1. Projected actual and discounted modular cash flows, with all cells having the same conversion result.

	All Module Cells Having Conversion Result -					
	Excellent		Fair		Poor	
Year	Projected Cash Flow	Discounted Cash Flow	Projected Cash Flow	Discounted Cash Flow	Projected Cash Flow	Discounted Cash Flow
0	$-189,200	$-189,200	$-189,200	$-189,200	$-189,200	$-189,200
1	70,000	63,636	56,000	50,909	35,000	31,818
2	-6,900	-5,702	-5,520	-4,562	-3,450	-2,851
3	52,500	39,444	42,000	31,555	26,250	19,722
4	52,500	35,858	42,000	28,687	26,250	17,929
5	52,800	32,785	42,240	26,228	26,400	16,392
6	33,900	19,136	27,120	15,309	16,950	9,568
7	25,600	13,137	20,480	10,509	12,800	6,568
8	30,900	14,415	24,720	11,532	15,450	7,208
9	30,700	13,020	24,560	10,416	15,350	6,510
10	30,700	11,836	24,560	9,469	15,350	5,918
Net Present Value (10%)		$48,364		$851		$-70,418

decisions made accordingly. There is sufficient time for the sequential conversion of two modules, if so desired, before the final decision must be made. A listing of all possible decisions is provided in Table 2. A decision to initiate a module conversion at the end of year 1, instead of year 0, will have a lower initial cost ($184,200 versus $189,200), but also results in the 10th year's cash flow being omitted by the planning horizon. End of year 2 initiations are treated similarly. For simplicity, we will only address the abandonment alterative as the decision to halt the conversion process at its then-current state, realizing that mixed modes of production procedures will be the result.

The possibility of incremental alternatives raises an important point. The advantages of sequential decision making have been discussed in the literature [Bierman and Rao (1978) and Cyert and DeGroot (1987)], and many corporations follow these practices for their research and development projects [Cook and Rizzuto (1989)]. However, these corporations generally have not adopted advanced manufacturing systems in an incremental fashion. These decisions are predominantly handled on an "all or do-nothing" basis. The repeatability structure and timely post-audit information requirements has limited the number of situations where the sequential approach is appropriate, and departmental parochial interests may have further reduced these opportunities [Gurnani (1984) and Mukherjee and Henderson (1987)]. Still, incremental alternatives are important, and we will use them in our presentation to highlight their potential.

Table 2. All possible decisions considered by the corporation.

Notation*	Decision
d_{00}	Do not convert any modules, reject this project.
d_{01}	Convert all six modules immediately.
d_{02}	Convert one module, collect data, and reevaluate this proposal in one year. (Given that this decision has been selected, the subsequent decisions follow.)
d_{10}	Stop the conversion process with one module.
d_{11}	Convert the remaining five modules.
d_{12}	Convert one more module, collect data, and make final decision at the end of year two. (Given that this decision has been selected, the subsequent decisions follow.)
d_{20}	Stop the conversion process with two modules.
d_{21}	Convert the remaining four modules.

*Note- First subscript numeral is the decision time and the second subscript numeral identifies a specific decision.

Preliminary Steps and Solution Notation. The classic net present value (NPV) term will refer to the discounted value of a particular strategy at the respective decision point of interest (time 0, end of year 1, and end of year 2). The term will be annotated by $NPV(MARR)_t$, with the subscript t referring to the decision time reference point.

For the next step, this paper will use the notation $EV(decision_j)_t$ to refer to the expected net present value of decision j at time t. This will be based on the most current probability distribution with the best alternative payoff. (This notation is used to simplify the decision tree branches in the subsequent figures.)

Decision trees will be developed, with systematic branching for events and decisions. In the figures, decision branches that are cross-hatched (—+—) are inferior decisions.

The cash flow projections in Table 1 are based on homogenous cell conversion results. Because the cells are expected to make near equal contributions to those cash flows, we can divide the module cash flows and NPV's by the number of cells to obtain cash flows and NPV's on a per cell basis. The NPV's for cell conversions initiated at time zero, end of year one, and end of year two, are listed in Table 3, by cell conversion result.

Since there are five cells per module, there are 21 possible combinations of cell conversion results for each module. By using the estimated $P(\Theta) = \{0.4, 0.4, 0.2\}$ in the multinomial probability function, we can find the probability associated with each combination. Table 4 lists all the outcome combinations, with the numbers of excellent, fair, and poor cell conversions (columns (a), (b), and (c),

Table 3. Cell net present values for different conversion initiation times.

Cell Conversion Result	Cell Conversion Initiation Time				
	Time 0	End of Year 1		End of Year 2	
	NPV $(10)_0$*	NPV $(10)_1$	NPV $(10)_0$	NPV $(10)_2$	NPV $(10)_0$
Excellent	$9,673	$8,306	$7,551	$6,702	$5,539
Fair	170	-723	-657	-1,807	-1,493
Poor	-14,084	-14,267	-12,970	-14,569	-12,041

*Note - NPV $(i)_t$ = net present value for interest rate i, at point in time t.

Table 4. All possible combinations of cell outcomes, for one module, including occurrence probabilities, net present values, and expected values.

Combination	Number of Cells Classified as			Probability of Combination	NPV $(10)_0$ for Combination	Expected Value of Combination
	Excellent (a)	Fair (b)	Poor (c)	(d)	(e)	(f)
1	5	0	0	0.0102	$48,364	$495
2	4	1	0	0.0512	38,862	1,990
3	4	0	1	0.0256	24,608	630
4	3	2	0	0.1024	29,359	3,006
5	3	1	1	0.1024	15,105	1,547
6	3	0	2	0.0256	851	22
7	2	3	0	0.1024	19,857	2,033
8	2	2	1	0.1536	5,603	861
9	2	1	2	0.0768	-8,651	-664
10	2	0	3	0.0128	-22,905	-293
11	1	4	0	0.0512	10,354	530
12	1	3	1	0.1024	-3,900	-399
13	1	2	2	0.0768	-18,154	-1,394
14	1	1	3	0.0256	-32,408	-830
15	1	0	4	0.0032	-46,661	-149
16	0	5	0	0.0102	851	9
17	0	4	1	0.0256	-13,402	-343
18	0	3	2	0.0256	-27,656	-708
19	0	2	3	0.0128	-41,910	-536
20	0	1	4	0.0032	-56,164	-180
21	0	0	5	0.0003	-70,418	-23

respectively), and the probabilities for each combination (column (d)). For a single module, we can use the NPV's associated with each cell outcome (Table 3) to find the NPV's corresponding to each combination (column (e)). Further, we can also obtain the combinations' expected values (column (f)). In fact, columns (e) and (f), correspond to terminal decision, d_{10}, values. Similar NPV computations can be obtained for decisions d_{01}, d_{11}, d_{20}, and d_{21}, as shown in Table 5.

Potential Gains from Incremental Automation with Perfect Post-audit Information. Because incremental methods have been relatively ignored in automation decisions, we will briefly review the benefits of that process, as applied to this study, before addressing the information updating procedures.

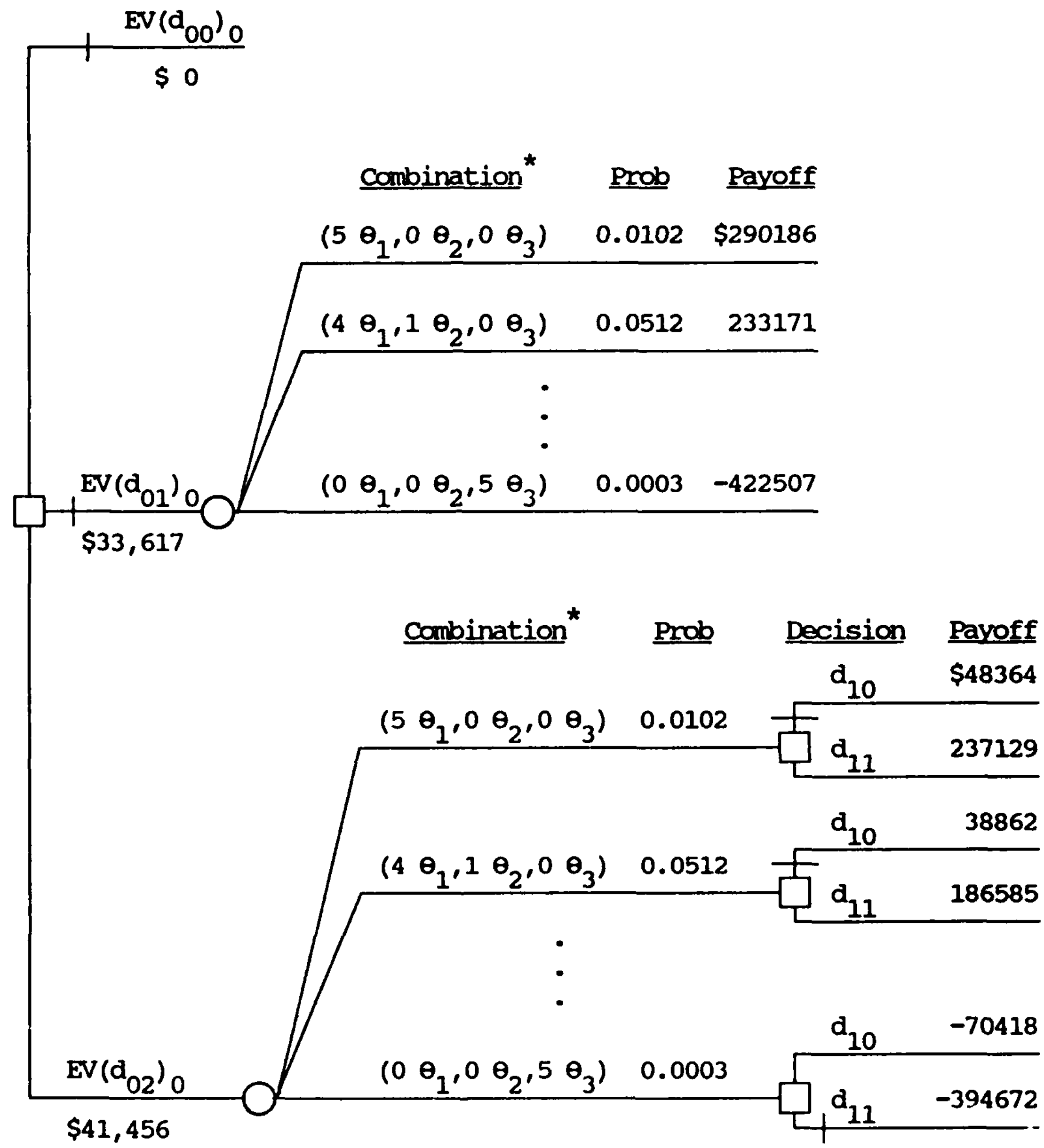
For the moment, we will hypothesize a situation where a sampled module provides perfect insight information into all other module conversions. Whatever results appear in the cells of a given sample module will also appear in the cells of all other modules. The decision alternatives are do nothing, d_{00}, convert all modules, d_{01}, or convert one module, d_{02}, and decide to convert the remaining modules or halt at the end of year 1 (decisions d_{11} and d_{10}, respectively). With this perfect information, there is no need for a second sample module. This leads to the development of the decision tree in Figure 5. The expected values summarized in the figure indicate that decision d_{02}, to sample one module before making the final decision, has the best payoff ($41,456 compared to $33,617 and $0). If the converted module has cell combinations 1 through 5,7,8, or 11, then the firm should convert all remaining modules, otherwise, they should halt the process. This is

Table 5. Net present values for terminal decisions, discounted to time zero, for cell combinations in one module - all modules having identical outcome combinations.

	NPV $(10)_0$* for Terminal Decisions				
Combination - Cell Results**	d_{01}	d_{10}	d_{11}	d_{20}	d_{21}
1. (5,0,0)	$290,186	$48,364	$237,129	$86,117	$196,889
2. (4,1,0)	233,171	38,862	186,585	68,406	151,051
3. (4,0,1)	147,647	24,608	110,769	41,840	82,295
4. (3,2,0)	176,155	29,359	136,041	50,695	105,213
5. (3,1,1)	90,632	15,105	60,224	24,129	36,457
6. (3,0,2)	5,109	851	-15,592	-2,437	-32,299
7. (2,3,0)	119,140	19,857	85,496	32,985	59,376
8. (2,2,1)	33,617	5,603	9,680	6,418	-9,381
9. (2,1,2)	-51,907	-8,651	-66,136	-20,148	-78,137
10. (2,0,3)	-137,430	-22,905	-141,952	-46,714	-146,893
11. (1,4,0)	62,124	10,354	34,952	15,274	13,538
12. (1,3,1)	-23,399	-3,900	-40,864	-11,293	-55,218
13. (1,2,2)	-108,922	-18,154	-116,680	-37,859	-123,975
14. (1,1,3)	-194,445	-32,408	-192,496	-64,425	-192,731
15. (1,0,4)	-279,968	-46,661	-268,312	-90,991	-261,487
16. (0,5,0)	5,109	851	-15,592	-2,437	-32,299
17. (0,4,1)	-80,414	-13,402	-91,408	-29,003	-101,056
18. (0,3,2)	-165,938	-27,656	-167,224	-55,570	-169,812
19. (0,2,3)	-251,461	-41,910	-243,040	-82,136	-238,569
20. (0,1,4)	-336,984	-56,164	-318,856	-108,702	-307,325
21. (0,0,5)	-422,507	-70,418	-394,672	-135,269	-376,081

Notes: * NPV $(i)_t$ = net present value for interest rate i, at point in time t.

** Cell results refers to number of cells having excellent, fair, and poor conversion results.



*Note - the combination values inside the parentheses refer to numbers of cells having outcomes θ_1, θ_2, θ_3. The topmost combination for each decision has five θ_1 cell outcomes, and zero θ_2 or θ_3. The next combination has four θ_1, one θ_2, and zero θ_3 outcomes, and so forth.

Figure 5. Advantage of the sequential alternative (d_{02}) over "all" (d_{01}) or "nothing" (d_{00}) alternatives, in a one sample, perfect information situation.

readily seen in a comparison of the combination NPV's for decisions d_{10} and d_{11}, in Table 5.

This shows the risk reduction provided by the sequential approach. In an earlier work [Bierman and Rao (1978)], this method was specifically proposed for divisible projects whose "all" alternative had a negative NPV. The sequential approach, as shown by this situation, can also be better than the "all" alternative for a project with a positive NPV. Here, the sequential option has a greater NPV then either the "all" or "do-nothing" alternative.

Gains from Incremental Automation with Imperfect Post-audit Information. The perfect sample information mentioned in the previous section rarely, if ever, occurs in real life. The cash flow realizations that are collected in years 1 and 2 are observed samples. Our questions are: (1) "How well does the sample data represent the true population?", and (2) "How well can the organization process and interpret this information?" The imperfect representation and interpretation characteristics make it necessary to establish conditional probability relationships between the apparent results and the true states of nature. These relationships could be based on historical record, subjective assessment, or a combination of many other factors.

The absence of historical performance records and the large number of cell result combinations (21), makes the determination of conditional assessments unwieldy. However, we can make it workable if we reduce the number of categories. We do this by grouping cell combinations to form module conversion categories. Using the cell

combination NPV's in Tables 4 and 5 as a guide, combinations 1-5 and 7 are grouped to form the "profitable" module conversion category, combinations 6, 8, 9, 11, 12, and 16 are grouped to form the "break even" category, and combinations 10, 13-15, and 17-21 are grouped to form the "unprofitable" category. The categories will be annotated as Φ_j, and $P(\Phi) = \{0.3942, 0.4198, 0.1859\}$ will be the initial module prior belief.

The establishment of the module outcome probabilities allows us to develop the conditional relationships. For this study, we will assume that the conditional probabilities listed in Table 6 represent the best beliefs for results from one year's sampling. Further, since this problem has two levels of sampling observations (a second module is sampled before the final automation decision is made), it is necessary to have a set of second level conditional probabilities. These conditional event probabilities are dependent upon both the second and the first module's apparent results. Therefore, the second level conditional probabilities in Table 7 are also assumed to be reasonable.

Before we can construct our decision tree, we need to compute the NPV's associated with each modular terminal decision, for the respective module conversion outcomes. (Recall that the cash flows in Table 1 were for homogenous cell conversion results.) Because our module conversion result is the summation of select cell combinations, we can easily obtain these values by using the following relationship, for each modular terminal decision, by conversion category:

$$\text{(module outcome probability)}*\text{(module payoff for decision } d_j\text{, given } \Phi_k) = \Sigma \text{ (weighted cell combination NPV's)} \qquad (7)$$

Table 6. Conditional probabilities for one year of sample data.

	Conditional Probabilities*		
True conversion result	$P(s_1 \mid \Phi_k)$	$P(s_2 \mid \Phi_k)$	$P(s_3 \mid \Phi_k)$
Φ_1	0.8	0.15	0.05
Φ_2	0.1	0.8	0.1
Φ_3	0.05	0.15	0.8

*Note - $P(s_r \mid \Phi_k)$ = probability of having a report, s_r, given that the true module conversion result is Φ_k, and

s_1 = module conversion reported as excellent,
s_2 = module conversion reported as fair, and
s_3 = module conversion reported as poor.

Table 7. Second level conditional probabilities.

Second module true result	Previous reported result	Conditional Probabilities*		
		$P(s_{11} \mid s_r, \Phi_k)$	$P(s_{12} \mid s_r, \Phi_k)$	$P(s_{13} \mid s_r, \Phi_k)$
Φ_1	s_1	0.95	0.05	0.0
	s_2	0.35	0.6	0.05
	s_3	0.05	0.85	0.1
Φ_2	s_1	0.4	0.55	0.05
	s_2	0.05	0.9	0.05
	s_3	0.05	0.6	0.35
Φ_3	s_1	0.05	0.85	0.1
	s_2	0.05	0.6	0.35
	s_3	0.0	0.05	0.95

*Note - s_{1t} = tth apparent result in the module converted in year 1

$P(s_{1t} \mid s_r, \Phi_k)$ = probability that apparent result s_{1t} will be observed in the second module sampled, given that s_r was observed in the first module, and Φ_k is the true state of nature of the second module.

where,

weighted cell combination NPV's

= (combination probability * combination NPV),
summing by module outcome category, by
terminal decision, and

module outcome probability = Σ cell combination probabilities, for a
particular module outcome category.

We can then rearrange equation (7) as:

$$(\text{module payoff for decision } d_j \mid \Phi_k) = \frac{\Sigma \text{ (weighted combination values)}}{\Sigma \text{ Combination probabilities}}$$

For example, terminal decision d_{10}, with a profitable outcome, has a payoff of

$$\begin{aligned}(d_{10} \mid \Phi_1) &= ((0.0102*48364) + (0.0512*38862) + (0.0256*24608) \\ &\quad + (0.1024*29359) + (0.1024*15105) \\ &\quad + (0.1024*19857)) \,/\, (0.3942) \\ &= \$24{,}608\end{aligned}$$

Thereby, the NPV's of all <u>module</u> terminal decisions are listed in Table 8.

<u>Time 0 Solution (Using Bayesian Techniques)</u>. To solve this problem for its initial decision strategy, it is recognized that there will be terminal decisions at time 0, end of year 1, and end of year 2. Those decisions that are terminal at time 0 (d_{00} and d_{01}) have directly computable expected values, based on the prior probability distribution and appropriate payoffs, and their values are \$0 and \$33,617, respectively. For those strategies involving sample information (d_{02} and d_{12}), it is necessary to construct the "nature's" tree, which shows the probabilistic branching of events and the conditional branching for

Table 8. Net present value of module terminal decisions, at time zero.

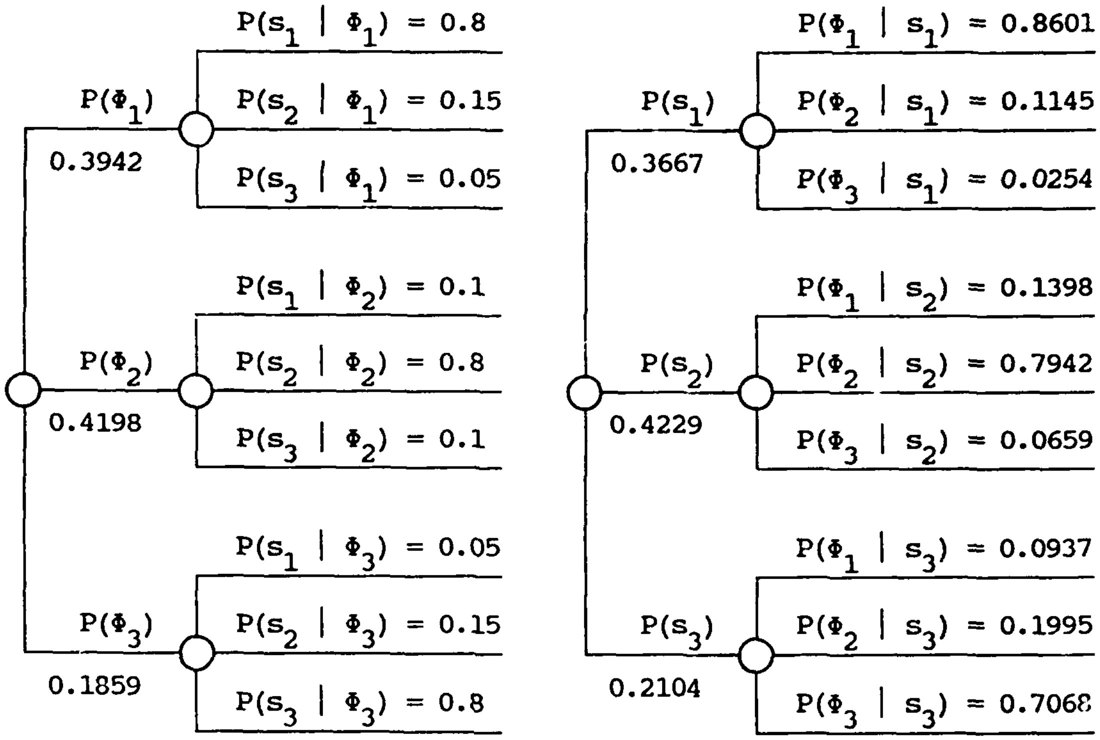
Terminal Decision	Conversion Result	Module $NPV(10)_0$
d_{00}	Φ_1, Φ_2, Φ_3	$ 0
d_{01}	Φ_1	147,647
	Φ_2	5,109
	Φ_3	-143,808
d_{10}	Φ_1	24,608
	Φ_2	851
	Φ_3	-23,968
d_{11}	Φ_1	110,769
	Φ_2	-15,592
	Φ_3	-147,606
d_{20}	Φ_1	41,840
	Φ_2	-2,437
	Φ_3	-48,696
d_{21}	Φ_1	82,295
	Φ_2	-32,299
	Φ_3	-152,021

reports, given those events. This "natural" branching is based on the prior distribution for the initial module and the imperfect nature of the sample information. The tree is revised to have the branches follow the order of the reported results, branching initially for s_r, and subsequently for s_{1t}. The revision process is an application of Bayes' theorem to convert from a prior belief with sample likelihood to a posterior belief and preposterior analysis, as shown in Figure 6. The joint probabilities are computed and then used to determine the marginal probabilities of reported results, for example:

$$\begin{aligned}
P(\Phi_1) * P(s_1 \mid \Phi_1) &= 0.3942 * 0.8 = 0.3154 \\
P(\Phi_2) * P(s_1 \mid \Phi_2) &= 0.4198 * 0.1 = 0.0420 \\
P(\Phi_3) * P(s_1 \mid \Phi_3) &= 0.1859 * 0.05 = \underline{0.0093} \\
& 0.3667 = P(s_1)
\end{aligned}$$

where,

$P(s_1)$ = Probability of getting an excellent conversion report. Similarly we compute that $P(s_2) = 0.4229$, and $P(s_3) = 0.2104$. Then, using the reported results from cell one as the initial branches, the conditional probabilities for the true states of nature, given the apparent results, are determined. This revision process is illustrated in Figure 6.(b). The revised tree is then used with the NPV's for end of year 1 terminal decisions to begin the construction of the decision tree in an "extensive" form analysis (fully detailing all branches). The expected values associated with terminal decisions at the end of year 1 are determined from the partially constructed decision tree in Figure 7. For remaining figures, the branches that terminate with decisions d_{10} and d_{11} will be abbreviated to carry only the $EV(d_j)_t$.



(a) Nature's Tree (b) Revised Tree

*Definitions: s_1 converted module appears to have profitable results
s_2 converted module appears to have breakeven results
s_3 converted module appears to have unprofitable results

Figure 6. Bayesian revision process for imperfect information in year one. (Prior beliefs and likelihood function converted to posterior beliefs and preposterior analysis structure.)

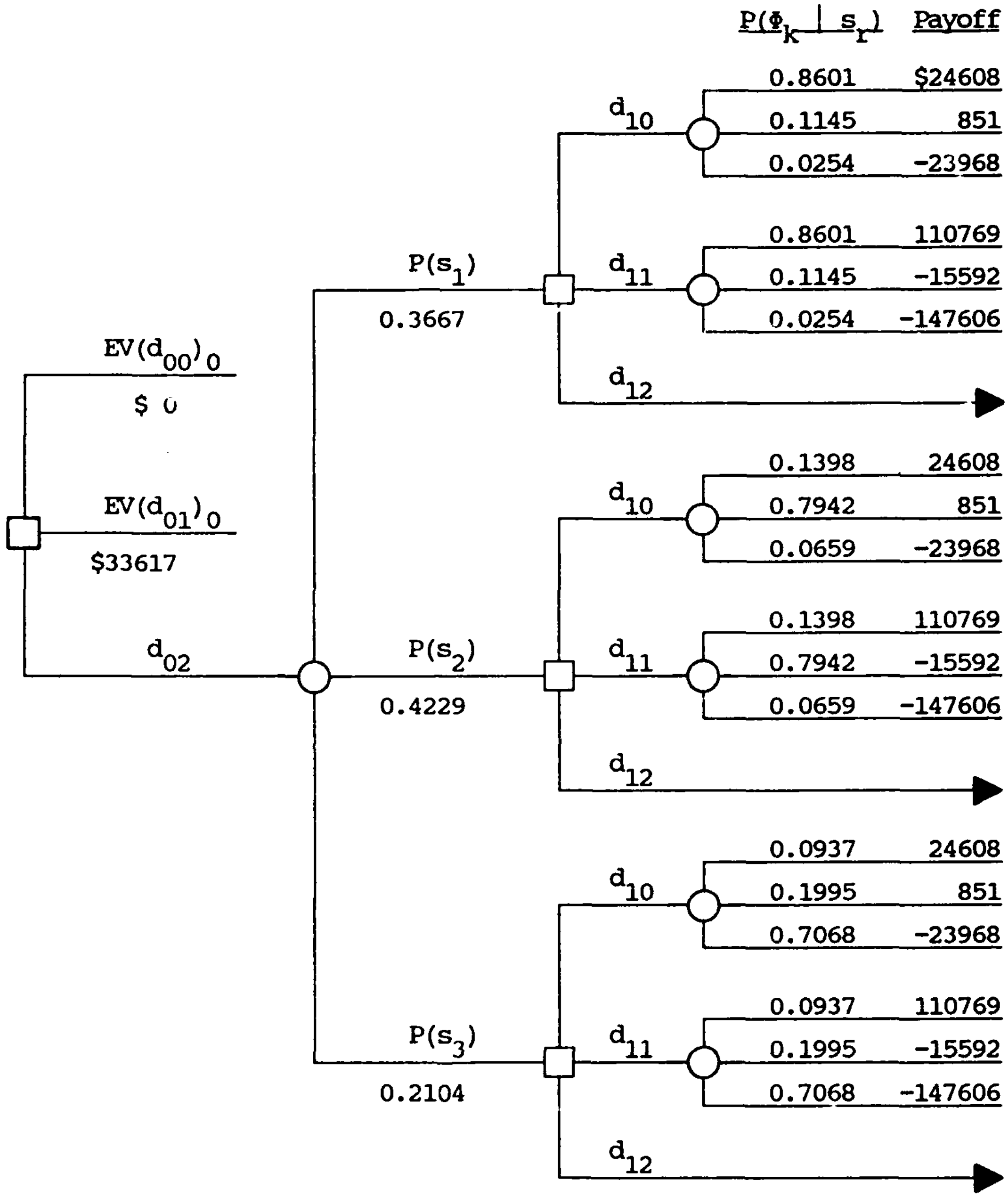
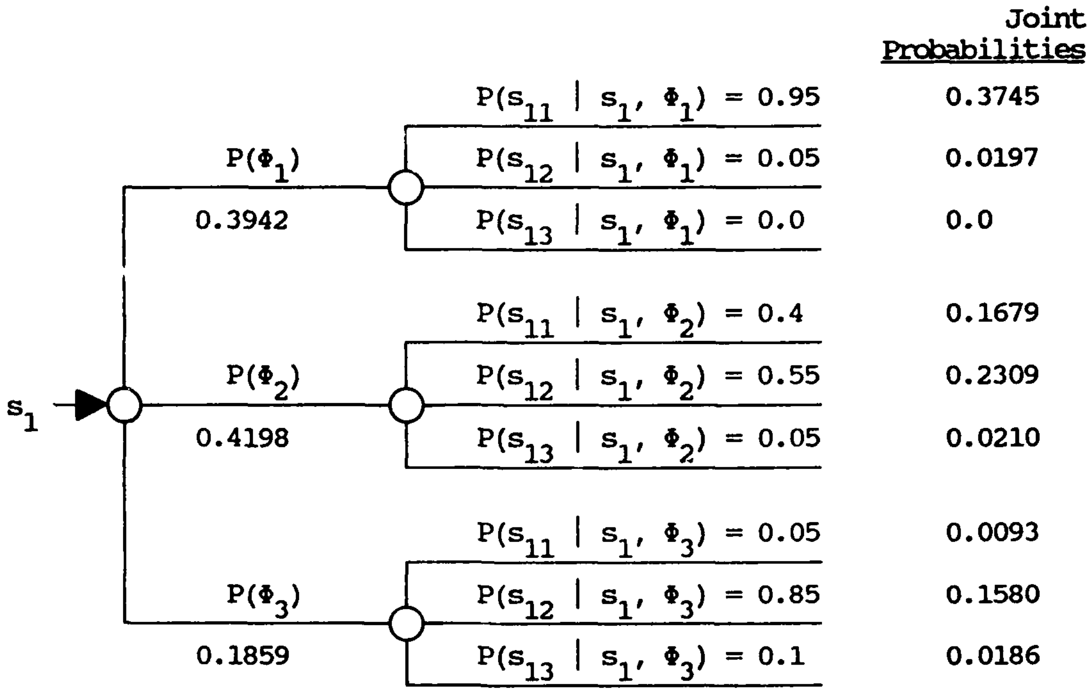
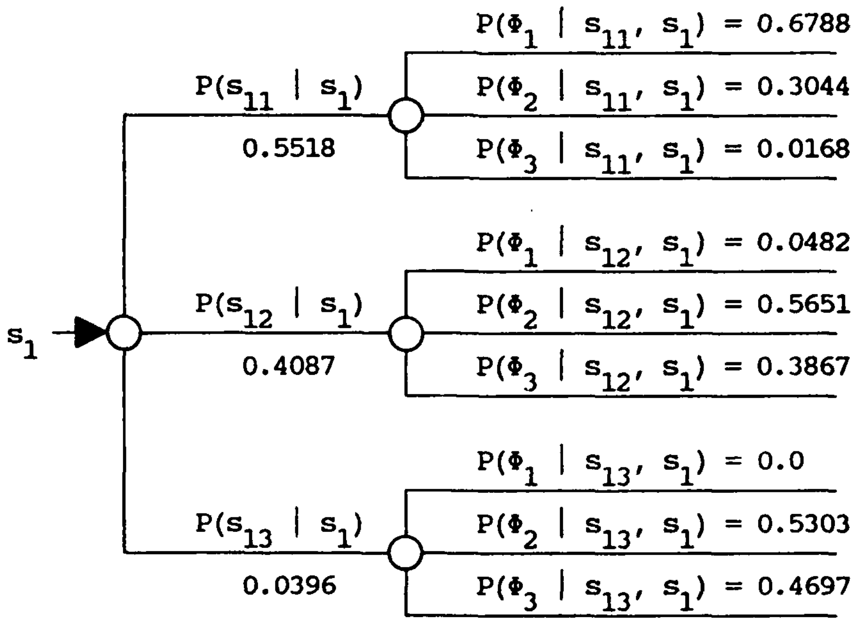


Figure 7. Partially constructed decision tree, with payoffs for strategies that are terminal at end of year one. (Branches for all d_{12} decisions will be developed subsequently.)

The three branches for decision d_{12}, appearing in each of the three major subbranches of Figure 7, require further expansion by the set of second level conditional probabilities. In each of these subbranches, the probability of the second module's report is dependent upon its true conversion result and the reported result of the first converted module. Figure 8 shows the "nature's" tree and the subsequently revised tree for the $P(s_1)$ branch. The revision procedures are again applied to find the probabilities of Φ_1, Φ_2, and Φ_3, given that the observable events s_{11}, s_{12}, or s_{13} have occurred (and the report from module 1, in this case, s_1). In Figure 8, given that s_1 has already been observed, the conditional probabilities of $P(s_{11} \mid s_1)$, $P(s_{12} \mid s_1)$, and $P(s_{13} \mid s_1)$ were found to be 0.5518, 0.4087, and 0.0396, respectively. Then, the conditional probabilities for the true conversion result, given s_1 and s_{1t} had occurred, were determined. (The steps illustrated in Figure 8 were then repeated for the $P(s_2)$ and $P(s_3)$ branches.) The conditional probability distribution defined at the end of year 2 is then used with the payoff values at the end of year 2 to complete the decision tree, and this extensive form analysis provides the initial decision strategy. Due to the detailed nature of the extensive form analysis, the final probability and payoff branches for all d_{20} and d_{21} decisions will be abbreviated as shown in Figure 9, to provide only the expected value of that decision. The use of this abbreviation technique is shown in Figure 10, the extensive form analysis for the initial decision strategy (a preposterior analysis). The initial strategy is:



(a) Nature's Tree



(b) Revised tree for s_1 branch.

Figure 8. Bayesian revision of s_1 branch for imperfect information in year two.

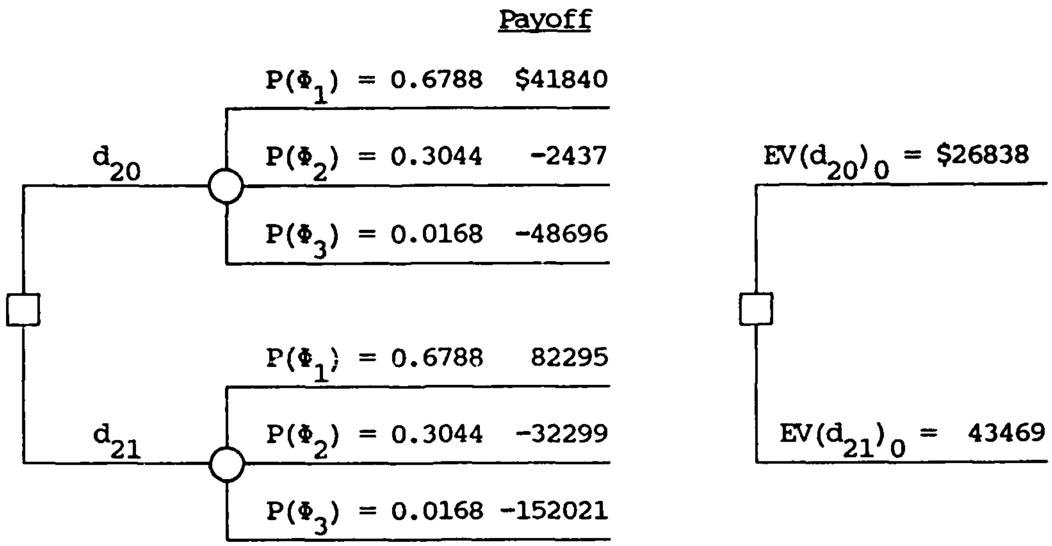


Figure 9. Sample simplification and annotation used for abbreviation of terminal branches of decision tree. Branch used for illustration is the one that follows observations s_1 in year one and s_{11} in year two.

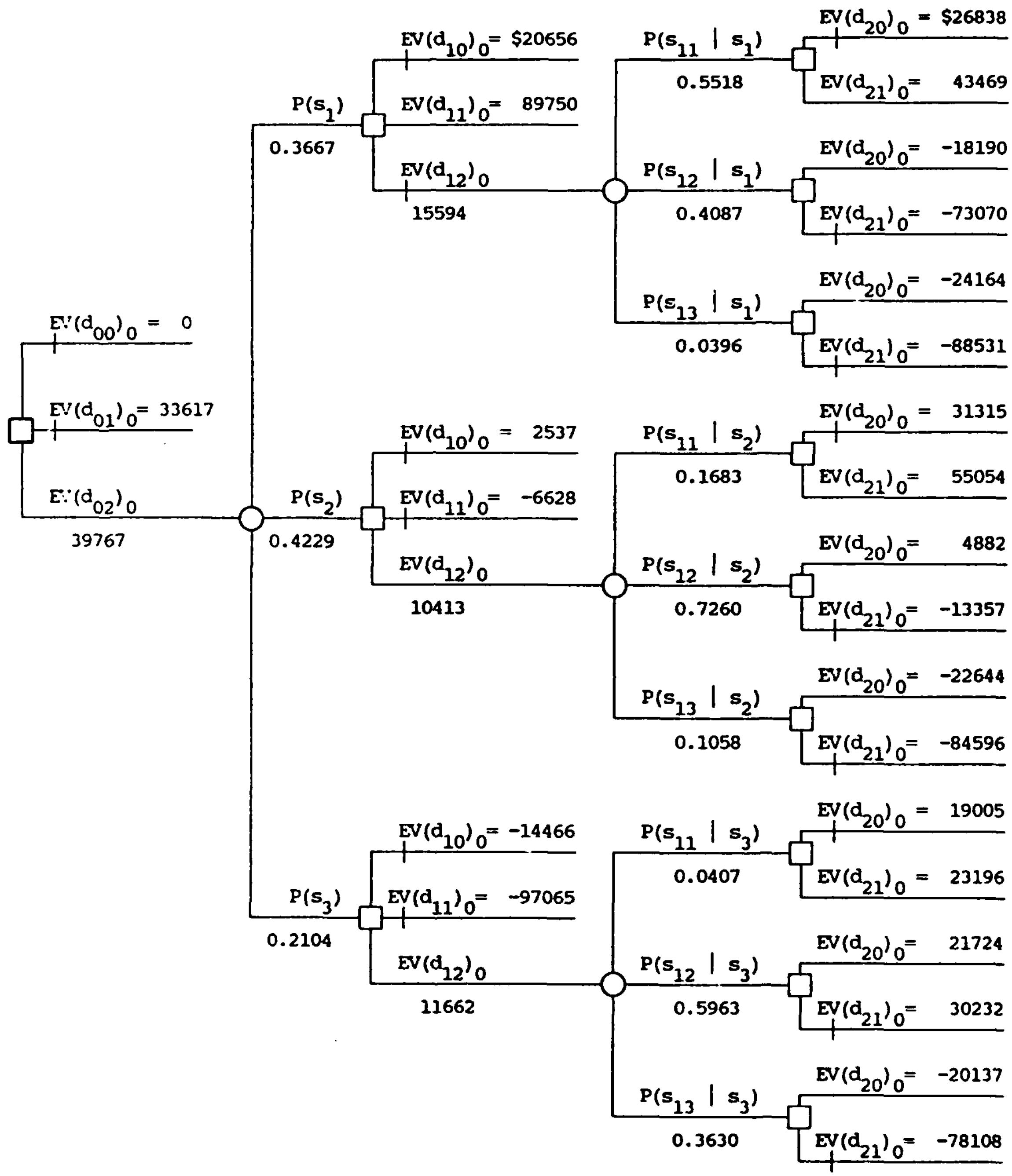


Figure 10. Extensive form analysis of decision tree at time zero when sample information is imperfect (preposterior analysis).

Time	Action
0	Convert a single module to the new equipment, review results, and make another decision at end of year 1 (decision d_{02}).
1	If module one has profitable results, then convert all five remaining modules (decision d_{11}). If module one has break even or unprofitable results, then convert a second module, review its results, and make final decision at end of year two (decision d_{12}).
2	a. If the first module result was break even and the second was profitable, then convert the remaining four modules (decision d_{21}). If the second module was break even or unprofitable, then terminate the process (decision d_{21}).
	b. If the first module result was unprofitable and the second was profitable or break even, then convert the four remaining modules (decision d_{21}). If the second module was unprofitable, then terminate the process.

The expected payoff from this strategy is $39,767.

STRATEGY REVISION BASED ON DIRICHLET-MULTINOMIAL MODEL

At the end of year 1, the first module's sample information is available, and the Dirichlet-multinomial model can be used.

Decision Strategy Revision. For this case study, we assume that the observed cellular conversion results are proportional to the true, but unknown, Dirichlet distribution of θ. Specifically,

Conversion result	True, but unknown probability	Converted module cells reporting this result
θ_1	0.2	1
θ_2	0.6	3
θ_3	0.2	1

Many firms would interpret this module's performance as falling into the break even category, and based on the strategies developed at time zero, would initiate the conversion of a second cell, with the final decision at the end of year two. However, a more efficient way to address this problem is to take the sample information, update the prior cell probability distribution, via the Dirichlet-multinomial model (equation (3)), develop revised module outcome probabilities, and then make decision adjustments. Following these steps, the cell outcome probabilities are revised as described by equation (5):

State of Nature	Prior θ_j	Initial α_j	Observed x_j	$\Sigma\alpha_j + x_j$	Posterior θ_j
θ_1	0.4	1.0	1	2.0	0.2667
θ_2	0.4	1.0	3	4.0	0.5333
θ_3	0.2	0.5	1	1.5	0.2
				7.5	

The posterior values are obviously more representative of the true, unknown, distribution (0.2, 0.6, 0.2) than the prior distribution.

The multinomial probability function then uses the cell posterior probabilities to determine the new probability values for the cell outcome combinations (as in Table 3, previously). These new combination probability values are grouped into the module outcome

categories to form the revised module outcome distribution, specifically, $P(\Phi) = \{0.2222, 0.4873, 0.2905\}$. The module terminal decision payoffs are updated by compounding the remaining pertinent values forward, to the end of year 1. These values are listed in Table 9. The revised module probabilities and terminal decision payoffs are used to reconstruct the decision tree. The expected values for each decision tree branch are computed by the methods shown previously. For example, decision d_{10} will have an expected value of

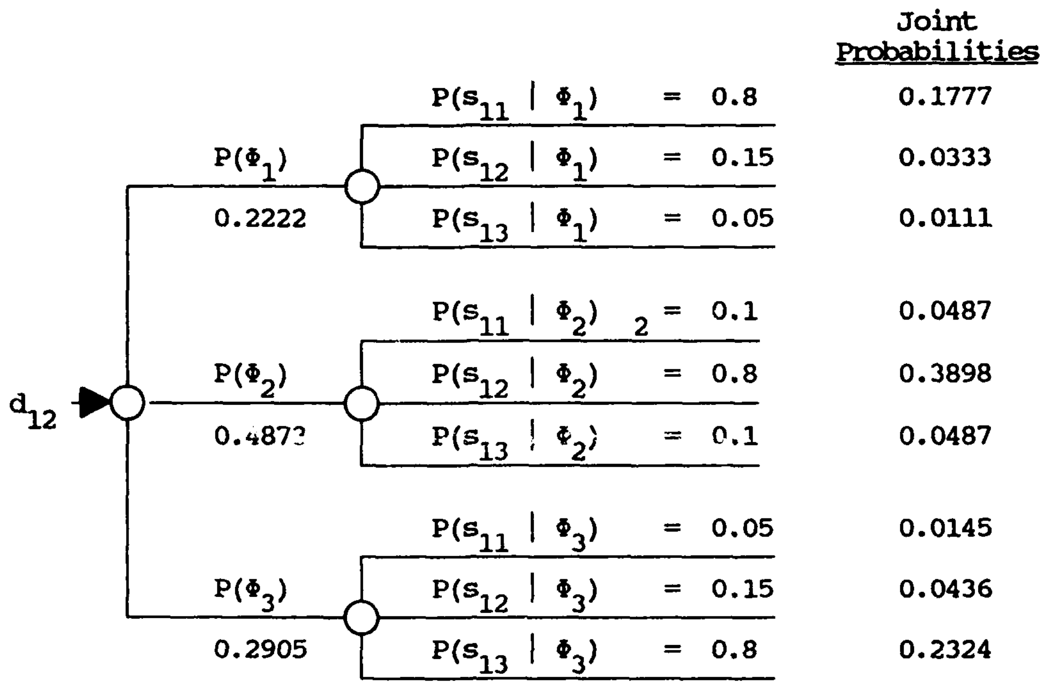
$$EV(d_{10})_1 = (0.2222 * 27069) + (0.4873 * 937) + (0.2905 * -26365)$$
$$= \$-1,190$$

and the expected value of decision d_{11} is similarly computed. Decision d_{12} will require another year of observations before its terminal decisions, d_{20} or d_{21}, can be made.

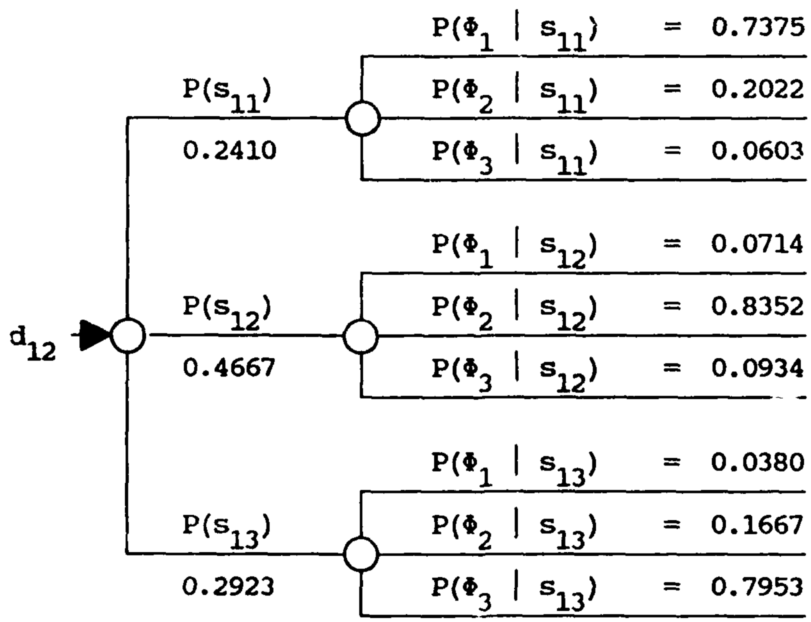
The revision is performed by using the new prior distribution in conjunction with the conditional probabilities for one level of sampling information. The conditional probabilities values used for $P(s_r \mid \Phi_k)$ in the time 0 analysis are now used for $P(s_{1t} \mid \Phi_k)$ in the end of year 1 analysis. Figure 11 illustrates the "nature's" and revised trees for the d_{12} branch. The nature's tree initially branches under the new prior distribution (0.2222, 0.4873, 0.2905), and then follows the one level conditional branching. The revision yields a preposterior analysis that is based on the anticipated apparent results of a second module's sampling. The revised probability tree is used to develop the extensive form analysis shown in Figure 12. Based on this revised analysis, the decision strategy is changed to terminate the conversion process with one cell (decision d_{10}).

Table 9. Net present values of remaining terminal decisions, discounted to the end of year one.

Terminal Decision	Conversion Result	Module NPV $(10)_1$
d_{10}	Φ_1 Φ_2 Φ_3	$ 27,069 937 -26,365
d_{11}	Φ_1 Φ_2 Φ_3	121,845 -17,151 -162,367
d_{20}	Φ_1 Φ_2 Φ_3	46,024 -2,681 -53,565
d_{21}	Φ_1 Φ_2 Φ_3	90,524 -35,529 -167,224



(a) Nature's Tree



(b) Revised tree for d_{12} branch.

Figure 11. Bayesian revision of d_{12} branch for imperfect information in year two, after sample information in year one has been used to revise prior probability distribution.

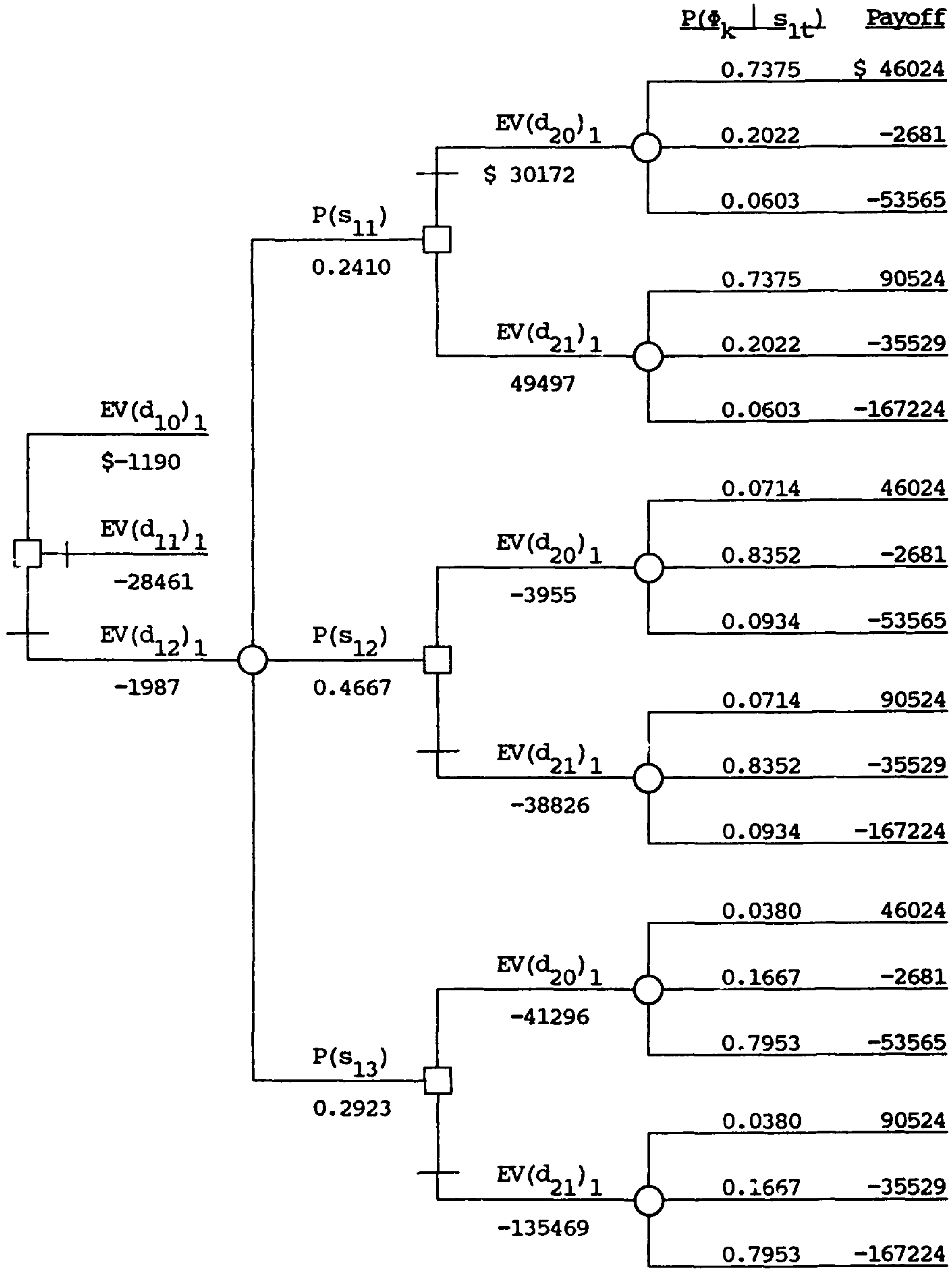


Figure 12. Decision tree for imperfect information example after sample information from year one has been incorporated. (Posterior analysis for year 1, and, simultaneously, preposterior analysis for year 2.)

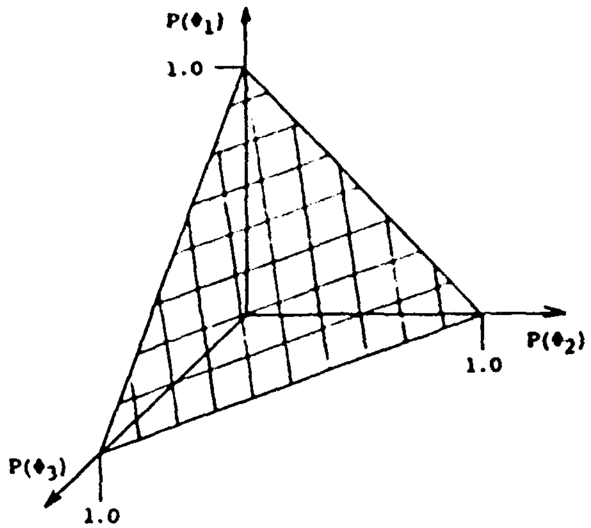
In summary, the expected payoff associated with the revised automation strategy is $-1,190. This expected payoff is considerably less than was anticipated at time 0. However, that expected payoff ($39,767) was based on the initial set of probabilities, a set that had an associated uncertainty that has now been somewhat resolved. The value of that initial decision strategy, at time 1, is now revised to be $-1,987. The value of the change in decision strategies is a savings of $797, a quantifiable measurement.

Another way to view this result is to consider the aspects of the expected value of sample information approach [Raiffa (1970), and Park and Sharp-Bette (1988)]. Consider, at time 0, that the expected payoff without sampling (decision d_{01} dominates d_{00}) is $33,617, and that the expected value with sampling is $39,767. The expected value of sample information is $6,150 (= $39,767 - $33,617), and, since the information is free, it is obvious that the sampling approach is the best option. At the end of year 1, the expected values of the halt conversion strategy and the further sampling approach are $-1,190 and $-1,987, respectively. At this point in time, the expected value of sample information is $-797 (=$-1,987 - ($-1,190)). Because the value of further sampling is negative, it is best to halt the conversion process.

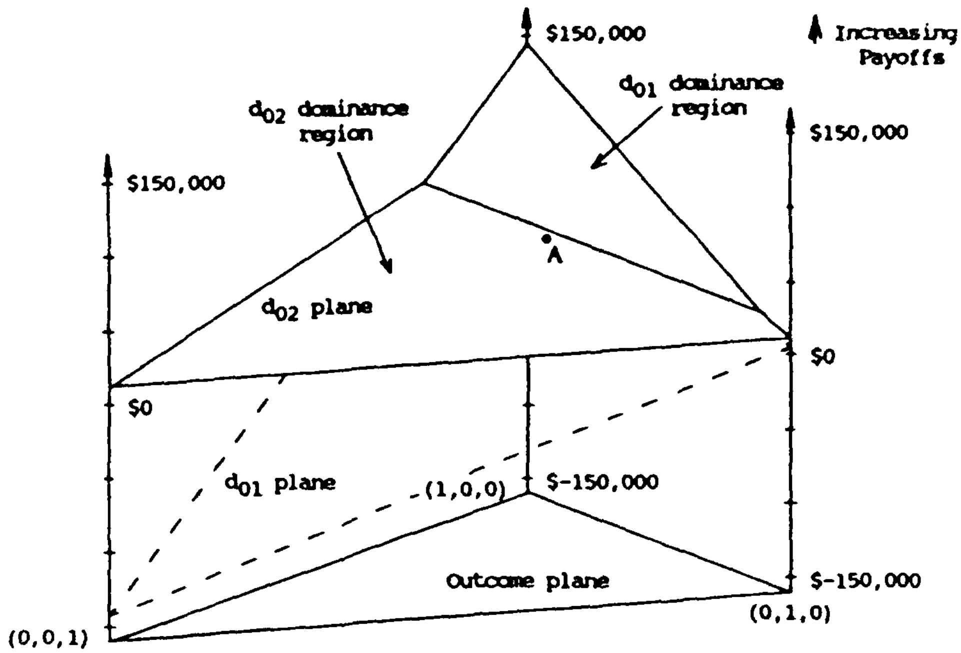
Sensitivity Analysis and Normal Form Analysis. For simplicity, the sensitivity analysis of the traditional factors described in the model development will not be performed. For any real problem, these analyses are extremely important. There is, however, a non-traditional

element that needs to be examined, and that is the initial descriptive shape parameter α. If the initial proposal had more historical data, then the α_j's could have been proportionately increased, to reflect a stronger prior belief. If the initial vector had instead been α = (6.0, 6.0, 3.0), then the posterior cell probability vector would have been P(θ) = (0.35, 0.4, 0.25), and the module vector would have been P(Φ) = (0.3296, 0.4499, 0.2204). This would have resulted in expected payoffs of $3,533 and $11,163 for d_{10} and d_{12}, respectively (decision d_{11} is dominated by d_{10}), and would have led to a continuation of the current decision strategy (sample in year 2, then make the final decision). This highlights the point that the initial shape parameter selections must be made very judiciously, and as objectively as possible.

The method of normal form analysis has been proposed as an alternative that provides additional insight when event probability assignment has a degree of associated uncertainty [Raiffa and Schlaifer (1961), Raiffa (1968), and Park and Sharp-Bette (1988)]. The thrust of the method is to find a dominant decision strategy over a range of probabilities. The cited references use a two outcome event example, with probabilities of θ and $(1 - \theta)$ for events one and two, respectively. This problem, having three possible results, requires an expansion of the previously cited method. Figure 13.(a) shows that if there are three outcome possibilities, the solution space of all possible probabilistic outcome combinations forms a triangular shaped plane. For each decision strategy, it is possible to construct a payoff response surface projecting from that probabilistic solution



(a) Plane representing all possible combinations of reported outcomes.



(b) Payoff planes and dominance regions for active decision strategies at time 0. Point A = (0.3942, 0.4198, 0.1859).

Figure 13. Case study normal form analysis at time zero.

space that varies according to the respective outcome combinations. For this problem, at time zero there are three active decision strategies (d_{00}, d_{01}, and d_{02}). The payoffs that correspond to each strategy's potential conversion results are

Strategy	Payoff for outcome Φ_1	Φ_2	Φ_3
d_{00}	$ 0	$ 0	$ 0
d_{01}	147,647	5,109	-143,808
d_{02}	89,750	10,413	11,662

Strategy d_{02} dominates d_{00}, as all of d_{02}'s payoffs are greater than d_{00}'s for the various outcomes. The construction of the payoff planes is shown in Figure 13.(b), with the respective dominance regions marked by solid lines. Solution by this method requires finding the decision payoff plane that has the highest elevation above the prior probability outcome combination (0.3942, 0.4198, 0.1859), or, if so desired, region of outcome combinations.

When the problem moves to subsequent decision points, payoff planes can be constructed for each relative decision strategy. The dominant strategy is then found for the revised outcome combination (like at the end of year 1, when $P(\Phi) = (0.2222, 0.4873, 0.2905)$). Thus, this generalization of normal form analysis can be used to support the other results.

CONCLUSIONS

This case study demonstrates how direct application of post-audit information can have a serious effect on a company's automation

strategy. When project estimates are developed as probability distributions having descriptive parameters that are not known with certainty, the availability of post-audit information creates the opportunity for uncertainty resolution to occur in those estimates, through revision of the distribution parameters. The gains that can be obtained are quantifiable, and the magnitude of that gain will be a direct reflection of the amount of resolution obtained.

In this case study, the decision to make all conversions at time zero (d_{01}) had a positive NPV. Under many corporations' current practices, they would have implemented a full transition. However, this positive NPV is derived from uncertain probabilities. In a sequential replacement process, the Dirichlet-multinomial model shows how these probabilities can undergo uncertainty resolution, and that our proposed automation decision process has the sensitivity and flexibility to permit decision strategy changes. For this study's conditions, the results of a full changeover are disastrous, when compared with our model's results.

The drawback to this method is that it requires detailed understanding of the potential sensitivity/insensitivity of the initial prior distribution to revision by sampling information. The initial parameter selection must consider the quality of prior information and its relative merit when compared with the quality and quantity of the incoming sample information. Still, as shown by our method, decision strategy modification, based on post-audit information, should increase company investment return and flexibility if it is incorporated into a company's capital budgeting plan.

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IV. PHASED CAPACITY EXPANSION - USING CONTINUOUS DISTRIBUTIONS TO MODEL PRIOR BELIEFS

ABSTRACT

Many current equipment replacement/production capacity expansion decisions carry increasing amounts of performance uncertainty when the alternative processes include technical innovations. In a sequential decision making environment, post-audit information can be used to resolve this uncertainty. In this paper, we demonstrate how three-point, PERT-type estimates can be used to determine continuous prior probability distributions that incorporate the post-audit information in Bayesian revision. We develop the concept of an equivalent sample size that initially is used to reflect our belief in the quality of the prior estimates, and subsequently in an assessment of the typicality of the observed sample data. A case study of an actual decision problem is used to illustrate the concepts.

INTRODUCTION

The capital budgeting process can be generalized into four basic areas: (1) alternative identification, (2) cash flow development and estimation, (3) project selection, and (4) post-audit and control. Because capital budgeting has historically been viewed as an

acquisition/allocation process, its literature has been dominated by discussions of comparison and selection methods [Moore and Chen (1984)]. This emphasis has led many businessmen to mistakenly believe that once they initiate a project, they are committed to it for its estimated life. The less emphasized area of post-audit and control focuses on monitoring the performance of implemented projects, and permits businessmen to update the initial estimates with observed results. If the revised estimates indicate that the project will not meet its expectations, the firm may want to abandon the project and reallocate its capital [Kee and Feltus (1982)]. Many businessmen also have the misconception that projects can only be taken on an "all or nothing" basis. Frequently, projects may be divided into several smaller, identical modules. The results from a few implemented modules can be used as essentially "free" sample information for the remaining modules. If the revised estimates are better than the initial ones, the firm can make additional replications, and profits can be made from projects that were initially unattractive [Bierman and Rao (1978), and Cyert and DeGroot (1987)].

Decision making for equipment replacement and production capacity expansion are areas that have not used this phasing methodology. Most replacement analyses have held a very limited scope, trying to solve the question: Should we keep the present machine (called defender) for an additional time period, or should we replace it with a new machine (called challenger)? Capacity expansion decisions have been limited to determining: How many more machines, like those currently being used, are needed to meet requirements? While surveys report that cash flow

estimation is the most difficult capital budgeting task, they also report that replacement decisions usually have the most accurate estimates [Pohlman, Santiago, and Markel (1988)]. This accuracy has led some businesses to solve these problems with deterministic methods, primarily using dynamic programming approaches [Oakford and Lohmann (1984), Bean, Lohmann, and Smith (1985), and Lohmann (1986)]. However, many current equipment replacement/capacity expansion decisions no longer fit this mold. The pursuit of a competitive advantage is leading more firms into newer, high technology manufacturing techniques that include computers, robotics, artificial intelligence, and flexible manufacturing cells, to name a few. These innovations broaden the decisions from a comparison of specific machines to a comparison of entire processes. While these challenger processes have increased potential, they do not have the situation-dependent performance histories of more traditional processes. Under this condition, where the challenger process is a substantial deviation from previous operations, the cash flow estimation accuracy no longer exists [Cook and Rizzuto (1989)]. We define the estimate inaccuracy that accompanies the new technology's cash flows as "uncertainty", which has a probabilistic nature with unknown descriptive parameters. As a new technology is implemented, the firms can employ the concept of uncertainty resolution to confirm/disprove their initial beliefs [Bierman and Hausman (1972)]. (For this process to be effective, the firm must provide a sequential decision making environment, with periodic continue/review decision points; otherwise, as the uncertainty

resolution occurs, the firm is unable to react, and it remains committed to the project.)

The generalized structure (initial estimate plus sample cash flow data yields a revised estimate) is the familiar Bayesian analysis framework. Many Bayesian models have been developed for specific probability distributions (normal, gamma, beta, etc.), but their complexities, and other probability concepts, have created some confusion for the businessmen who are supposed to use them [Koehler (1968)]. Some have reported difficulties in understanding the theoretical concepts and/or identification of hypothetical models that match real situations. In this paper, we will present the following:

1. Demonstrate how Bayesian analysis techniques can, and should, be applied to equipment replacement and capacity expansion problems that involve technological innovations. This will provide uncertainty resolution for the challenger process.
2. Provide an interpretation of a cash flow's 3-point, PERT-type estimates (optimistic, most likely, and pessimistic) that leads directly to the development of continuous distribution models.
3. Develop the concept of equivalent sample size for periodic cash flows. This permits us to take better advantage of the sample information (adjusting for anomalous or better than expected conditions), as capital budgeting problems involving equipment usually do <u>not</u> have large samples sizes.

4. Illustrate these techniques through the case study of an actual equipment replacement/capacity expansion decision problem.

In presenting these concepts, we are limiting the approach to using continuous distribution models that have natural conjugate properties.

MODEL DEVELOPMENT

Our procedure starts with Bayes' theorem for continuous probability models. To briefly review that theorem, we are given an uncertain continuous quantity of interest, θ, with beliefs about its occurrence behavior that are summarized as a probability density function. The belief that exists before sample information is taken is called the prior belief. If we describe the sample information involving θ by the statistic X, then the parameter's posterior density is the conditional density of θ, given $X = x$, written as:

$$f(\theta \mid X = x) = \frac{f(\theta, x)}{f(x)}$$

where,

$f(\theta \mid X = x)$ is the conditional density of one random variable, given a second random variable,

$f(\theta, x)$ is the joint distribution of the two random variables,

$f(x)$ is the marginal density of the second random variable.

The joint density, $f(\theta,x)$, and the marginal density, $f(x)$, are usually not known, but, they are expressible in terms of the prior distribution and the sample likelihood function. Defined as:

$$f(\theta, x) = f(\theta) * f(x \mid \theta)$$

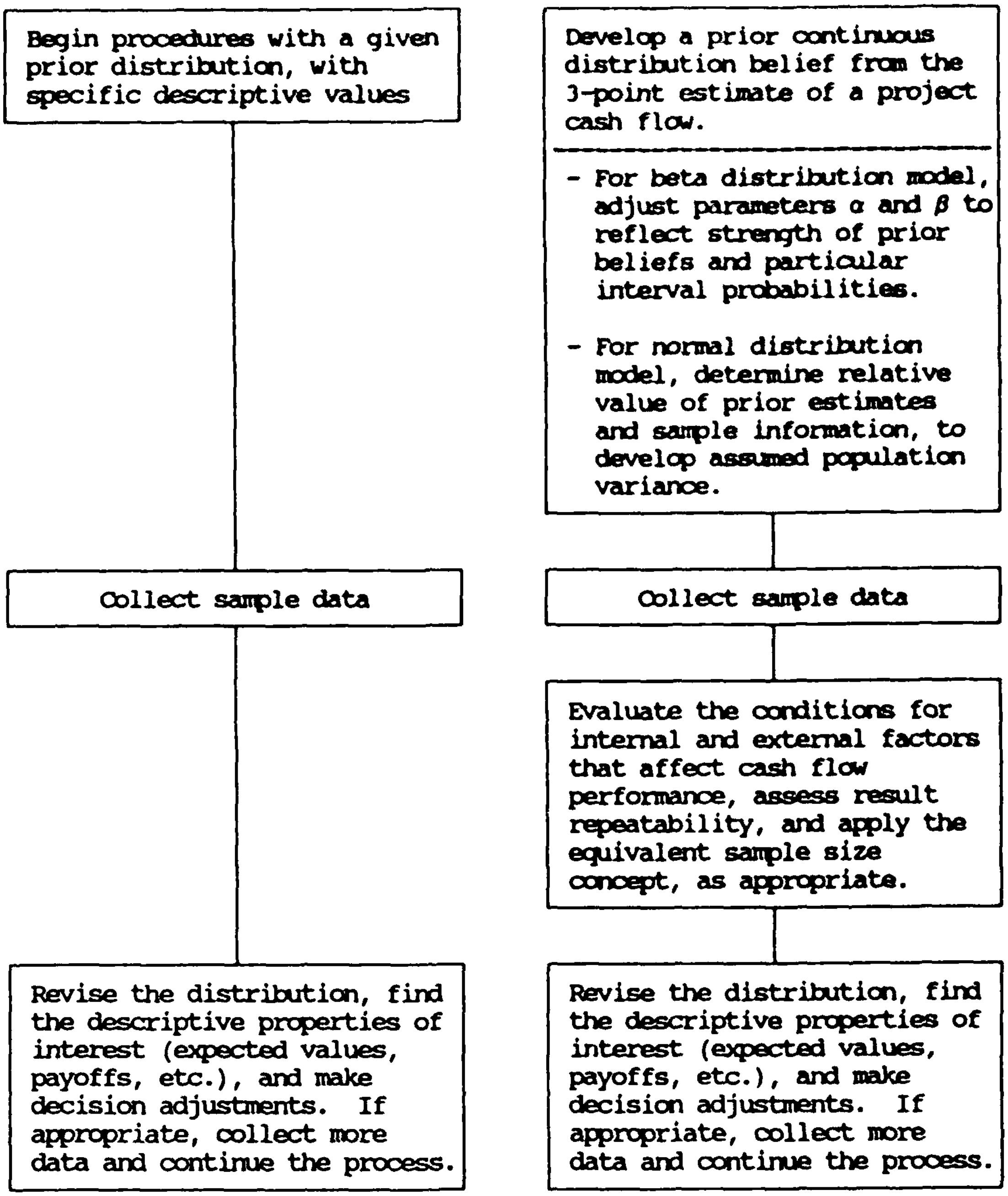
$$f(x) = \int_{-\infty}^{\infty} f(\theta) * f(x \mid \theta)\, d\theta$$

so that,

$$f(\theta \mid x) = \frac{f(\theta) * f(x \mid \theta)}{\int_{-\infty}^{\infty} f(\theta) * f(x \mid \theta)\, d\theta}$$

which provides the revision structure. Also, the marginal distribution of x, in the denominator, is also referred to as the sample's predictive distribution. Conventional Bayesian methods are listed in Figure 1.(a).

Bayesian models have been elegantly developed for many probability distributions, but few firms use probability theory to describe their cash flow estimations, more typically using single values (the deterministic approach), or 3-point, PERT-type estimates [Pohlman, Santiago, and Markel (1988)]. Some analysts are reluctant to fit a smooth distribution to these estimates, because they feel the proposals do not provide enough information about the distribution of events and/or their own lack of familiarity with certain distributions. As an alternative, they fit the 3-points to a triangular distribution, particularly where repeated computer simulation runs can be used for analysis [Lohmann (1986)]. Therefore, our first proposed modification is to provide a means where the 3-point estimates can be readily fit to a continuous probability distribution, so we can take advantage of the available Bayesian methods. Subsequently, we will also address the concept of equivalent sample sizes, once data has been obtained. Our modified procedure is outlined in Figure 1.(b).



(a) Conventional method.

(b) Modified method.

Figure 1. Comparison of conventional Bayesian methods and the modified approach.

<u>Beta distribution</u>

As stated previously, 3-point estimates are frequently used to develop triangular distributions, but, as shown in Appendix A, the cumulative density functions (c.d.f.'s) of the triangular distribution and certain families of beta distributions have almost indistinguishable differences, making them probabilistically interchangeable. Therefore, we can use the beta distribution to model the predicted outcomes, providing a more descriptive model, and a distribution with natural conjugate properties. The standardized beta density function is:

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\ \Gamma(\beta)}\ x^{\alpha-1}\ (1 - x)^{\beta-1} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

with,

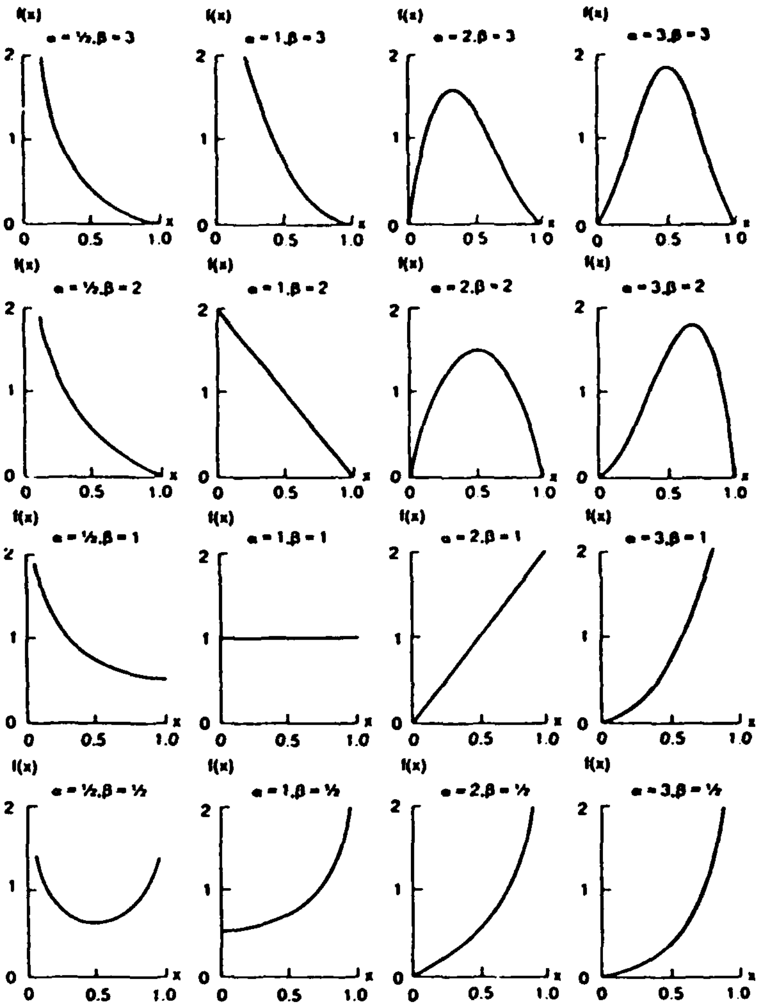
$$\text{mean} = \alpha/(\alpha + \beta) \tag{1}$$

$$\text{mode} = (\alpha - 1)/(\alpha + \beta - 2) \quad \text{, which also equals } x_m \tag{2}$$

$$\text{variance} = (\alpha\beta)\ /\ ((\alpha + \beta)^2(\alpha + \beta + 1)) \tag{3}$$

and its shape can be easily changed by varying the shape parameter values (α and β), as shown in Figure 2. Further, if α and β are greater than zero, the distribution density function touches the horizontal x-axis at x=0 and x=1. These intersections imply that the end points established by the optimistic and pessimistic estimates are absolute limits. If they are poorly chosen, and a sample falls outside of these limits, the prior density function (P(x)=0 for x not between 0 and 1) will cause the posterior probability for that value to remain zero. We will use the following estimate notation:

A = interval lower limit = pessimistic estimate



$$
f(x;\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, & 0<x<1;\alpha>0;\beta>0 \\ 0, & \text{otherwise} \end{cases}
$$

Figure 2. Examples of beta distributions with various shape parameter sets (α, β).

M = value of interval mode = most likely estimate

B = interval upper limit = optimistic estimate

and, the standardized terms are

a = lower limit = 0

x_m = value of the standardized mode = (M - A)/(B - A)

b = upper limit = 1

Initial prior belief. Conventional Bayesian methods for beta distributions can only begin once an initial belief is given. However, when the information is provided as a 3-point estimate, the initial belief must first be derived. Thereby, if we use the PERT assumption that the variance = $(1/6)^2$, then equations (2) and (3) can be solved simultaneously for α, giving

$$(\alpha - 1)^3 + (7x_m - 36x_m^2 + 36x_m^3)(\alpha - 1)^2 - 20x_m^2(\alpha - 1) - 24x_m^3 = 0 \qquad (4)$$

and this cubic equation (in terms of $(\alpha - 1)$) can be solved for the positive values of α. Then, with this α, equation (2) can be used to find the value of β. We can use α and β to find the cumulative density of any interval, C to D, on t, by using the nonstandard form of the incomplete beta function:

$$\frac{\displaystyle\int_C^D \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \left[\frac{t - A}{B - A}\right]^{\alpha-1} \left[\frac{B - t}{B - A}\right]^{\beta-1} dt}{\displaystyle\int_A^B \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \left[\frac{t - A}{B - A}\right]^{\alpha-1} \left[\frac{B - t}{B - A}\right]^{\beta-1} dt} \qquad (5)$$

and factoring common terms and canceling gives

$$\frac{\int_C^D (t - A)^{\alpha-1} (B - t)^{\beta-1}\, dt}{\int_A^B (t - A)^{\alpha-1} (B - t)^{\beta-1}\, dt} \tag{6}$$

An earlier work [Greer (1970)], also uses this approach, but we will deviate from that work, as we will now refine the initial belief by using the incomplete beta function to obtain more information about our derived probability distribution. If we are not satisfied with the cumulative probability for a specific range, we can adjust the values of α and β to obtain the desired value. However, the changes must be proportional, to maintain the value of x_m. For example, the sets (α,β) = (2,2) and (3,3), have differently shaped distributions (see Figure 2), but share a common mode, 0.5.

The proportional changes to α and β leave the mode unchanged, but creates changes the variance value. The intent of this variance term variation is a better portrayal of the prior belief. It does, however, create deviations from the well known PERT equations:

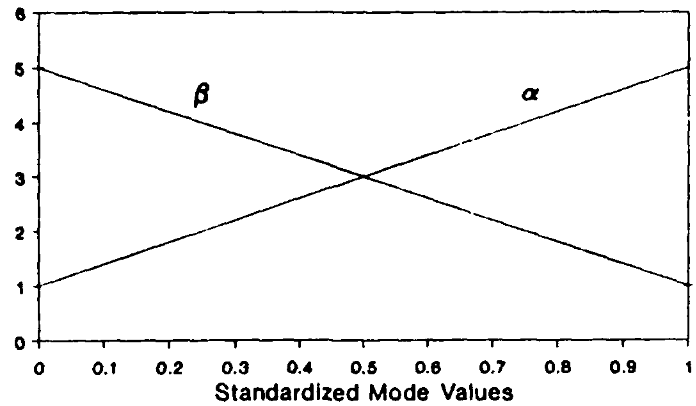
$$\text{mean} = (A + 4M + B)/6 \qquad \text{,for general distributions} \tag{7}$$

$$= (4M + 1)/6 \qquad \text{,for standardized distributions} \tag{8}$$

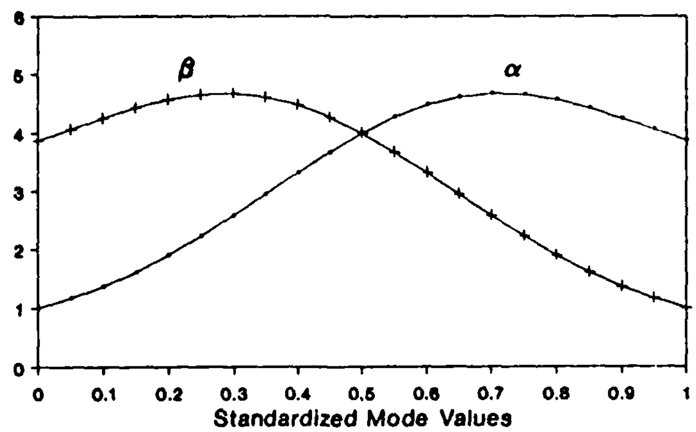
$$\text{variance} = ((B - A)/6)^2 \qquad \text{,for general distributions} \tag{9}$$

$$= (1/6)^2 \qquad \text{,for standardized distributions} \tag{10}$$

However, these formulas are just approximations, and holding strictly to them limits the user to a specific family of beta distributions [as seen in Greer (1970), Fielitz and Myers (1975), and Littlefield and Randolph (1987)]. An earlier work [Swanson and Pazer (1971)] shows that if we solve equations (1) and (8) simultaneously, we obtain the linearly related family of α and β in Figure 3.(a). If we solve



(a) Family of α and β values that satisfy the PERT and beta equations for the mean.



(b) Family of α and β values that satisfy the beta mode and variance equations, with the PERT approximation that the variance = $(1/6)^2$.

Figure 3. Graphs of families of α and β values that satisfy particular PERT and beta equations.

equations (2), (3), and (10) simultaneously, we obtain the family shown in Figure 3.(b).

To remove this constraint, we will only use the assumption, variance = $(1/6)^2$, for our <u>initial</u> α and β determinations. We will refine α and β, using insights from the incomplete beta function, realizing that these refinements will change the variance value. This provides more accurate values for the prior distribution mean and variance, because the values will be computed directly from the shape parameters and not from simplifying approximations.

An interactive BASIC computer program is attached at Appendix B, and it is designed to take the user from the proposal's 3-point estimate, to the initial prior beta distribution. After refinement, the program provides summary information about α, β, mean, mode, and variance. The program approximates the incomplete beta function by using Simpson's rule. Sampled values from the program have been compared to Pearson's <u>Tables for the Incomplete Beta Function</u> and have been found to be accurate to four decimal places.

<u>Equivalent sample size concept - in the initial prior belief</u>. Another purpose of parameter refinement is to permit us to subjectively "weight" our prior assessment. By doing so, we are introducing the concept of an equivalent sample size. (This differs from an earlier work [Smidt (1979), where the term referred to the conversion of a continuous variable to a dichotomous variable, in an investigation of biased decisions.) As we develop the initial parameters, we are making inferences about the quality of our estimates, effectively assigning our prior beliefs an "equivalent" sample size, n_e. This sample size is

the sum of α and β. If we use proportionately large values of α and β, we are implying that we have a strong belief about the mode's value. Conversely, if we use smaller values, we are expressing a weak belief. In using this equivalent sample size concept, we must make a relative worth assessment, considering the quality of the estimate and its relationship with the amount and timing of the sample information. For example, if the estimates are felt to be as good as 20 of the coming observations, then $n_e = 20$. (In a totally uninformed prior information condition, $\alpha = \beta = 1$, which corresponds to a uniform distribution. In this condition, the distribution of the sample information will also be the posterior distribution.)

This method has increased flexibility over previous works because it is not constrained by the PERT variance assumption. It will also provide more accurate values for the prior distribution mean and variance because those values are computed directly from the shape parameters, and not from simplifying approximations, such as PERT equations (6) and (8) or improved PERT equations like the extended-Pearson-Tukey approximations [Keefer and Bodily (1983)] (which can be applied to a variety of distributions). The use of the incomplete beta function will also permit more accurate distribution fitting than previous methods that fit the moments of the distribution to fractiles and quartiles [Pratt, Raiffa, and Schlaifer (1965)].

<u>Conventional Bayesian methods for beta distributions</u>. If a beta distribution describes the prior beliefs, then a Bernoulli process, either Binomial or Pascal sampling, is needed to keep within a natural conjugate framework. (In an equipment replacement situation, it is

unlikely that a firm would set a specific number of successes, and continue investing until they attained that number. Therefore, we will focus on the binomial sampling approach.) A Bernoulli trial has only two outcomes, described as "success" or "failure". However, since cash flow data is a continuous variable and not a dichotomous one, we will satisfy the Bernoulli description by categorizing the cash flow as a success if it exceeds its most likely estimate, and a failure if it does not. The shape parameters are transformed to this success/failure orientation by

$$r' = \alpha \tag{11}$$

$$n' = \alpha + \beta \tag{12}$$

where the (') denotes a prior belief, and equations (1) through (3) become:

$$\text{mean} = r' \,/\, n' \tag{13}$$

$$\text{mode} = (r' - 1) \,/\, (n' - 2) \tag{14}$$

$$\text{variance} = (r'(n' - r')) \,/\, (n'^2(n' + 1)) \tag{15}$$

Data is collected as r successes in n trials. The prior belief and likelihood function form the natural conjugate family. This relationship is readily proven by:

beta prior * binomial likelihood

$$= \frac{f(x \mid n', r') * l(x \mid n, r)}{\int_0^1 f(x \mid n', r') * l(x \mid n, r)\, dx}$$

$$= \frac{\frac{\Gamma(n')}{\Gamma(r')\,\Gamma(n' - r')}\, x^{r'-1}\, (1 - x)^{n'-r'-1}\, \frac{n!}{r!\,(n-r)!}\, x^{r}\, (1 - x)^{n-r}}{\int_0^1 \frac{\Gamma(n')}{\Gamma(r')\,\Gamma(n' - r')}\, x^{r'-1} (1 - x)^{n'-r'-1}\, \frac{n!}{r!\,(n-r)!}\, x^{r}\, (1 - x)^{n-r}\, dx}$$

$$= \frac{x^{r'+r-1} (1 - x)^{n'-r'+n-r-1}}{\int_0^1 x^{r'+r-1} (1 - x)^{n'-r'+n-r-1} \, dx}$$

$$= \frac{x^{r'+r-1} (1 - x)^{n'-r'+n-r-1}}{\frac{\Gamma(r' + r) \, \Gamma(n' - r' + n - r)}{\Gamma(n' + n)}}$$

$$= \frac{\Gamma(n' + n)}{\Gamma(r' + r) \, \Gamma(n' - r' + n - r)} \, x^{r'+r-1} (1 - x)^{n'-r'+n-r-1}$$

= beta posterior

The parameters are revised by the established formulas (where the (") denotes a posterior belief):

$$n'' = n' + n \tag{16}$$

$$r'' = r' + r \tag{17}$$

Equivalent sample size concept - in the sample observations. While a theoretic Bernoulli trial has only two possible outcomes, "real life" data frequently lacks that absolute clarity. Consequently, before we incorporate the sample, we must consider: (1) "How well does the sample information represent the parent population?", (2) "How well can the firm interpret the reported results?", and (3) "What is the relative quality of the sample information when compared to the estimated performances?" The first two questions focus on result repeatability, and the third question examines the equitability of the distribution revision to be performed. We can resolve these questions by again applying the equivalent sample size concept.

A given period, n, has a cash flow, a_n, that is the average of all cells that have been implemented. The period also has optimistic, most likely, and pessimistic estimates that we will denote as a_{no}, a_{nm}, and

a_{np}, respectively. The value of the actual cash flow is related to the estimates by

$$a_n = a_{np} + (r_e/n_e)(a_{no} - a_{np}) \tag{18}$$

where r_e and n_e are the "equivalent" sample's number of "successes" and "trials". If the user does not desire to interject subjectively into the process, he can use $n_e = n$. But, if he feels that the representative quality of the data makes it an efficient performance indicator, he can increase the equivalent sample, such that $n_e > n$. This increases the "weight" of the sample information. Equations (16) and (17) are modified as

$$n'' = n' + n_e \tag{19}$$

$$r'' = r' + r_e \tag{20}$$

Conversely, if the user feels that conditions have made the sample an anomaly, he can make $n_e < n$. (Appendix C contains a model development for the normal distribution. It offers a unique derivation of prior beliefs, based on the 3-point estimates.)

Method summary

Before presenting the case study, we will summarize the overall method, and annotate (*) where our innovations are inserted.

1.* Develop a prior belief from a 3-point performance estimate, using a continuous distribution. Adjust prior distribution's descriptive parameters to reflect the strength of the prior belief, by using the equivalent sample size concept.

2. Given the prior belief and sampling to be performed, identify

all possible sample outcomes, and their predictive probabilities.

3. Determine the posterior distributions for each outcome possibility, and their respective expected values.

4. Determine the terminal decision expected values (applying the step 2 predictive probabilities to the step 3 expected values).

5. Formulate decision strategy.

6. Given that a phased implementation strategy is initiated, collect sample information.

7.* Evaluate sample and economic environment consistency (subjectively assess the repeatability of the sample information). Apply n or n_e, as appropriate, to obtain new posterior distributions.

8.* Determine new expected values and revise the decision strategy, if n_e has been used.

9. Continue this process throughout the project's economic life.

The conventional Bayesian approach is designed to provide a comprehensive decision strategy that will have a contingency plan for every possible event, and our modifications provide increased flexibility for those methods.

CASE STUDY: INCREMENTAL CAPACITY EXPANSION

The following case study is provided as illustration of the techniques presented. The study is an examination of an actual

capacity expansion and equipment replacement decision made by a large corporation.

General situation

The United Aerospace Corporation is a large manufacturer of military and commercial aircraft. These aircraft use five different sizes of engines, and their respective designs require a total of 84 different component airfoils (precision-forged compressor blades). Two years ago, the corporation built a highly automated 700,000 square foot facility, with a designed annual capacity of 1.3 million blades that supports a 10-year production requirement. After the factory was completed, two events occurred that created capacity problems for the firm. The first event, a favorable one, the corporation won an additional procurement contract from the government. This contract will increase the annual requirements by 500,000 blades within two years. The second event, an unfavorable one, was a problem in the finish application process. The airfoils have very fine tolerances for their finished surfaces and edges. Originally, these finishes were to be applied by a series of manual buffing operations (finish application was not an automated process). However, complications resulted in a near two-fold increase in buffing time, creating a bottleneck in the operations. The factory currently has eight manually operated buffing centers, but the complications and increased demands will require the factory to double its current finish application capacity within two years.

The corporation can meet its capacity expansion requirements by purchasing eight more manually operated buffing centers, or by adopting automated abrasive flow machines (AFM), a technology that has just become available. These machines use a synthetic putty, impregnated with silicon carbide, that flows under pressure across the edges and surfaces of the part. The pressured flow removes material and provides the desired finish. The flow is computer controlled and the process is fully automated. It is projected that one AFM has the capacity of four buffing centers, and two AFM's could meet the increased capacity requirements. As an extension of this problem, the corporation also considers replacement of their existing buffing centers with AFM's. This extension means that the corporation has three configuration alternatives to consider: (1) sixteen manually operated buffing centers, (2) four AFM's, or (3) some mixture of buffing centers and AFM's. However, there are concerns over the fact that these machines are a technical innovation for this process, and uncertainty accompanies their predicted performances.

An incremental approach is used to highlight the differences in the cash flow performances of the two technologies. One AFM currently costs $265,000 more than four buffing centers. The AFM is expected to have lower labor and operating costs (compared to four buffing centers), and its faster processing time is expected to permit reductions in the work-in-progress (WIP) levels for all types of blades, resulting in lower inventory holding costs. When the proposal was prepared, the incremental after-tax cash flow (one AFM - four buffing centers) was given in the form of optimistic, most likely, and

pessimistic estimates. These values are listed in Table 1 for the eight-year planning horizon (the dollar values that appear in the tables and on the figures are rounded to the nearest $10, and any apparent inconsistencies in calculated values are due to this round-off). The range of estimates is a reflection of the uncertainty associated with the AFM's performance. To address the concerns over this uncertainty, the AFM alternative has the option of phasing its machines into operation over the available window of two years. Additionally, the AFM's have computer components that will be cheaper in the near future. The equipment supplier has told the firm that the equipment's current list cost will be reduced by $15,900 at the end of the year, and by another $16,560 ($32,460 total) at the end of two years. Accordingly, the firm prepared per unit (one AFM to four buffing centers) incremental estimates of the after-tax cash flows for equipment initiated at the end of year 1, or the end of year 2. The latter cash flows are limited by the planning horizon, but the AFM's will have increased salvage values that correspondingly increase the final year's after-tax cash flow. This information is shown in Table 2. The incremental cash flows that have been thus far developed explicitly address the capacity expansion comparison of AFM to buffing center. For the equipment replacement consideration, the on-hand buffing equipment would normally have a lower value than new centers, and the cash flows would be adjusted accordingly. However, because these centers are both relatively inexpensive and readily adaptable to other factory needs, the value of the current equipment is felt to be comparable to new equipment, and we will assume that the incremental

Table 1. Estimated annual cash flows for one incremental unit (one AFM - four buffing centers) initiated at time zero.

	Estimate		
Year	Pessimistic	Most Likely	Optimistic
0	\$-265,000	\$-265,000	\$-265,000
1	34,200	38,000	39,900
2	60,300	67,000	70,350
3	63,000	70,000	73,500
4	61,200	68,000	71,400
5	58,500	65,000	68,250
6	58,500	65,000	68,250
7	58,500	65,000	68,250
8	58,500	65,000	68,250
$EV(u_{01})_0$*	-17,760	9,710	23,450

*Note - $EV(u_{01})_0$ refers to the expected value of a cash flow for one unit, initiated at time 0, discounted at the MARR of 15% (value rounded to the nearest \$10).

Table 2. Estimated annual cash flows for one incremental unit (one AFM - four buffing centers) initiated at the end of years one and two, discounted to the periodic decision times.

	Initiation time of Abrasive Flow Machine					
	End of Year 1			End of Year 2		
Year	Pessimistic Estimate	Most Likely Estimate	Optimistic Estimate	Pessimistic Estimate	Most Likely Estimate	Optimistic Estimate
0	$ 0	$ 0	$ 0	$ 0	$ 0	$ 0
1	-249,100	-249,100	-249,100	0	0	0
2	34,200	38,000	39,900	-232,540	-232,540	-232,540
3	60,300	67,000	70,350	34,200	38,000	39,900
4	63,000	70,000	73,500	60,300	67,000	70,350
5	61,200	68,000	71,400	63,000	70,000	73,500
6	58,500	65,000	68,250	61,200	68,000	71,400
7	58,500	65,000	68,250	58,500	65,000	68,250
8	70,500	81,000	86,250	87,500	102,000	109,250
$EV(u_{21})_2$ *				$-13,880	$ 12,490	$ 25,670
$EV(u_{j1})_1$	$-16,470	$ 10,380	$ 23,800	-12,070	10,860	22,320
$EV(u_{j1})_0$	-14,320	9,030	20,700	-10,490	9,440	19,410

*Note - $EV(u_{j1})_t$ refers to the expected value of a single unit initiated at time j, discounted at the MARR of 15%, to a time of interest, t, with the values rounded to the nearest $10.

cash flows for the AFM's to existing centers can be treated as the same as those developed in Tables 1 and 2.

The firm's after-tax minimum attractive rate of return (MARR) is 15-percent. The net present value for the incremental cash flows (AFM - buffing center) are interpreted by the following:

Positive net present value = AFM performance is superior,

Negative net present value = buffing center performance is superior.

To clarify the multiple options created by the phased adoption policy, a complete listing of all possible decisions is provided in Table 3. For this corporation, a decision that creates mixed modes of finish application is acceptable. (For brevity's sake, the sensitivity analysis of interest rates, projected savings, etc., will not be presented in this paper, but it recognized that such analyses are critical to any decision problem.)

Beta distribution model

Our first step is the selection of a beta prior probability distribution that will satisfactorily model our uncertainty in the cash flow's expected value. We will use the following notation:

$EV(u_{t1})_j$ = expected value of one unit (one AFM - four buffing centers), initiated at time t, discounted at the MARR to time j.

$EV(d_{jk})_q$ = expected value of terminal decision k, made at time j, discounted at the MARR to time q. This term combines unit expected values (as for, $EV(d_{01})_0 = 4 * EV(u_{01})_0$).

Table 3. Decisions, and follow-on options, that were considered at time zero.

Decision	Explaination
d_{00}	Do not initiate any units, reject this alternative.
d_{01}	Initiate all four units immediately.
d_{02}	Initiate one unit, collect data, and reevaluate this proposal in one year. (Given that this decision has been selected, the options continue.)
d_{10}	Stop the incremental process with one AFM, acquire four buffing centers to meet remaining capacity requirements.
d_{11}	Initiate the remaining three units.
d_{12}	Initiate one more unit, collect data, and make the final decision at the end of year two. (Given that this decision has been selected, the options continue.)
d_{20}	Stop the incremental process with two units. Evaluate if it would now (the end of year 2) be better to replace the existing AFM's with buffing centers.
d_{21}	Initiate the last two units, to replace the buffing centers that were installed when the factory was built.

Development of the prior distribution. We use the 3-point estimate for a single unit at time 0 (\$-17,760, \$9,710, \$23,450) to determine a standardized mode, x_m, and solve equation (4) to get α = 4.64, and equation (2) gives β = 2.82. The set (4.64, 2.82) describes our initial candidate beta distribution, which has a per unit expected value, $EV(u_{01})_0$ = \$7,870, and a probability of negative return, or the buffing centers' performance being superior to the AFM's, $P(u_{01} < 0)$ = 0.14. At this point, we propose that this candidate solution be refined, because our initial feeling is that the probability that the AFM's performance will not exceed the buffing centers' performance is greater than 0.14. After reviewing several other candidate parameter sets, we select α = 3.667, β = 2.333, with $E(u_{01})_0$ = \$7,420 and $P(u_{01} < 0)$ = 0.18. The selected parameters retain the mode at \$9,710, but the variance has changed from the equation (9) approximation of $6{,}870^2$, to a directly computed value of $7{,}590^2$ (through a non-standardization of equation (3)). The variance change is a direct reflection of our desired change in a particular interval's probability, depicting our prior beliefs better than the PERT approximations.

Conventional Bayesian methods at time 0. The remaining steps in the time 0 analysis follow conventional Bayesian techniques. Since they have not been typically considered in equipment replacement scenarios, those steps are presented in abbreviated fashion. The payoffs for the time 0 terminal decisions are (using (α, β) = (3.667, 2.333)):

$EV(d_{00})_0$ = \$0

$EV(d_{01})_0$ = \$29,690

Under a pure net present value criterion, the positive $EV(d_{01})_0$ would dictate that we initiate all units immediately. However, in our modified procedures, we follow the course of action that yields the highest expected value. So, continuing, the values of α and β are used in equations (11) and (12) to give r' = 3.667 and n' = 6. The incremental options available in this project permit the initiation of a single unit at time 0, with a second unit, if desired, at the end of year 1. The initiation of a single unit implies that our sample information will have n=1, with possible outcomes r=0 and r=1. An r=0 outcome means that the observed result falls short of the year 1 expected value, while an r=1 outcome means that the performance prediction was exceeded. The <u>predictive</u> probabilities for those outcomes, given n', r', and n, follow the beta-binomial distribution, with $P(r=0 \mid r',n',n) = 0.389$, and $P(r=1 \mid r',n',n) = 0.611$. The year one sample results for (r, n) = {(0, 1) and (1, 1)} are combined with r' and n' in equations (16) and (17) to yield two posterior sets of descriptive parameters, (r", n") = {(3.667, 7) and (4.667, 7)}. We then compute the per unit expected values and probabilities of loss for units initiated at time 0 and time 1. The results are:

r	r"	n"	$EV(u_{01})_0$	$P(u_{01} \leq 0)$	$EV(u_{11})_0$	$P(u_{11} \leq 0)$
0	3.667	7	\$3,830	0.31	\$4,020	0.27
1	4.667	7	9,710	0.10	9,030	0.08

From this, we compute the end of year 1 terminal decision payoffs:

for r = 0, $EV(d_{10})_0 = EV(u_{01})_0 = \$3{,}830$

$EV(d_{11})_0 = EV(u_{01})_0 + 3 * EV(u_{11})_0 = \$15{,}890$

for r = 1, $EV(d_{10})_0 = \$9{,}710$

$EV(d_{11})_0 = \$36{,}790$

The posterior values r" and n" then become the prior values for the second year's sampling. The predictive probabilities for year 2 are:

Year 1	Year 2			
r	r'	n'	r	$P(r \mid r', n', n)$
0	3.667	3.333	0	0.476
0	3.667	3.333	1	0.524
1	4.667	2.333	0	0.333
1	4.667	2.333	1	0.667

The per unit expected values for the end of year 2 distributions are (all values of r" have an associated value of n" = 8):

r''	$EV(u_{01})_0$	$P(u_{01} \leq 0)$	$EV(u_{11})_0$	$P(u_{11} \leq 0)$	$EV(u_{21})_0$	$P(u_{21} \leq 0)$
3.667	\$3,210	0.28	\$1,730	0.40	\$1,130	0.45
4.667	6,950	0.09	6,110	0.16	6,280	0.19
5.667	10,690	0.02	10,480	0.04	11,430	0.05

and the payoffs for the end of year 2 terminal decisions are:

Year 1 r	Year 2 r	r''	n''	$EV(d_{20})_0$	$EV(d_{21})_0$
0	0	3.667	8	\$2,860	\$9,280
0	1	4.667	8	12,390	26,280
1	0	4.667	8	12,390	26,280
1	1	5.667	8	21,910	43,290

Although the cash flow is continuous, the discrete nature of the sample information makes the problem readily presentable in decision tree form, as shown in Figure 4. (Branches that are cross-hatched, —+— , are inferior decisions.) The strategy determined at time 0 is to initiate single units at time 0 and the end of year 1, and initiate the remaining units at the end of year 2. The expected return for this strategy is \$30,060. The strategy mapping shown in Figure 4 illustrates the level of detail provided by the Bayesian methods. All conventional sample outcomes are anticipated and appropriate actions

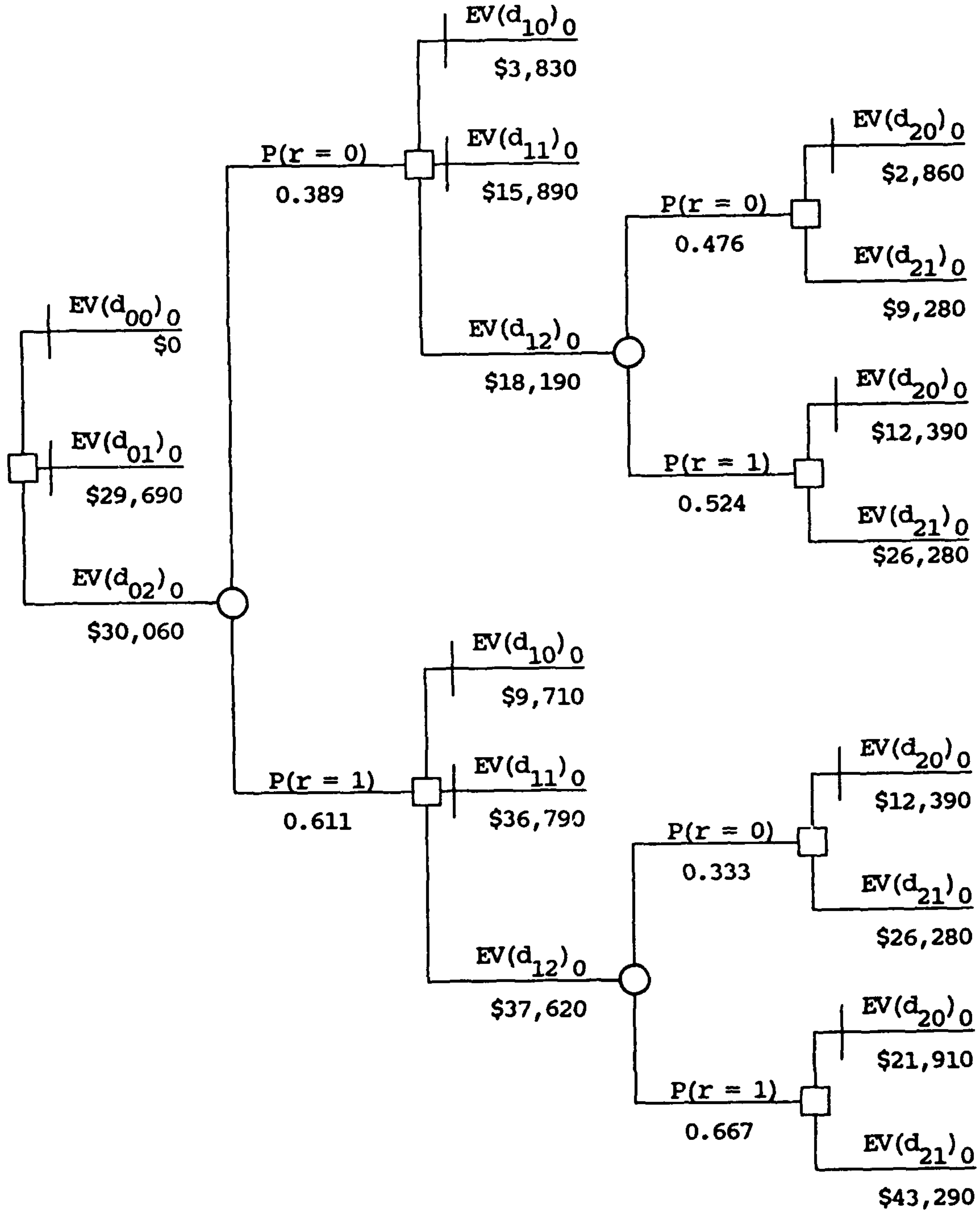


Figure 4. Extensive form analysis, at time zero, with predicted sample results for years one and two.

are enumerated for each possibility. The observations in years 1 and 2 will "steer" the movement along the extensive form analysis.

Applying equivalent sample size concept to the beta model. The year 1 incremental cash flow realization was $39,330, which was easily greater than the predicted value $38,000. The time 0 decision strategy would have the firm initiate a second unit, and continue monitoring. Here, we again deviate from the conventional approach. During year 1, the firm's economic environmental conditions were noted, and a review of the typicality, or repeatability, of the year 1 performance results was conducted. This review concluded that the sample data was as informative as the initial predictions, and should carry as much informative "weight" as those forecasts. Therefore, rather than use a value of $n = 1$ for the sample, we will use an equivalent sample size of $n_e = 6$. We then solve equation (18) for r_e:

$$r_e = (6 * (39,330 - 34,200)) / (39,900 - 34,200) = 5.4$$

This result is not possible under conventional methods. The values of r_e and n_e generate a new set of posterior parameters, $r'' = 9.067$ and $n'' = 12$.

Conventional methods for remaining end of year 1 analysis. We then use these parameters to resolve the decision problem. The per unit expected values that are used with end of year 1 terminal decisions are:

r''	n''	$EV(u_{01})_1$	$P(u_{01} < 0)$	$EV(u_{11})_1$	$P(u_{11} < 0)$
9.067	12	$15,380	0.01	$13,960	0.01

The payoffs for the end of year 1 terminal decisions are:

$EV(d_{10})_1 = \$15,380$

$EV(d_{11})_1 = \$57,260$

The beta-binomial predictive probabilities, with r' = 9.067, n' = 12, n = 1, for the second year's sampling are:

r	P(r | r', n', n)	r"	n"
0	0.244	9.067	13
1	0.756	10.067	13

Based on the posterior distributions, the per unit expected values are (all values of r" have associated values of n" = 13):

r"	$EV(u_{01})_1$	$P(u_{01} \le 0)$	$EV(u_{11})_1$	$P(u_{11} \le 0)$	$EV(u_{21})_1$	$P(u_{21} \le 0)$
9.067	$12,630	0.02	$11,620	0.02	$11,920	0.01
10.067	16,270	0.01	14,720	0.00	14,560	0.00

The payoffs for the end of year 2 terminal decisions are:

r"	n"	$EV(d_{20})_1$	$EV(d_{21})_1$
9.067	13	$24,250	$48,080
10.067	13	31,000	60,110

These results can also be mapped into a decision tree format, as shown in Figure 5. This extensive form analysis shows that the new optimal strategy is to convert all remaining units immediately. The expected return from this strategy is $57,260. The expected return, as compared to the time 0 return of $30,060, has nearly doubled. The net gain from the strategy change is small ($57,260 - 57,170 = 90), but the magnitude of that gain will always be situation and performance dependent. Further, the sample information shows that the probability of the AFM's performance exceeding the buffing centers is greater than we initially anticipated. The accelerated implementation decision, d_{11}, could only

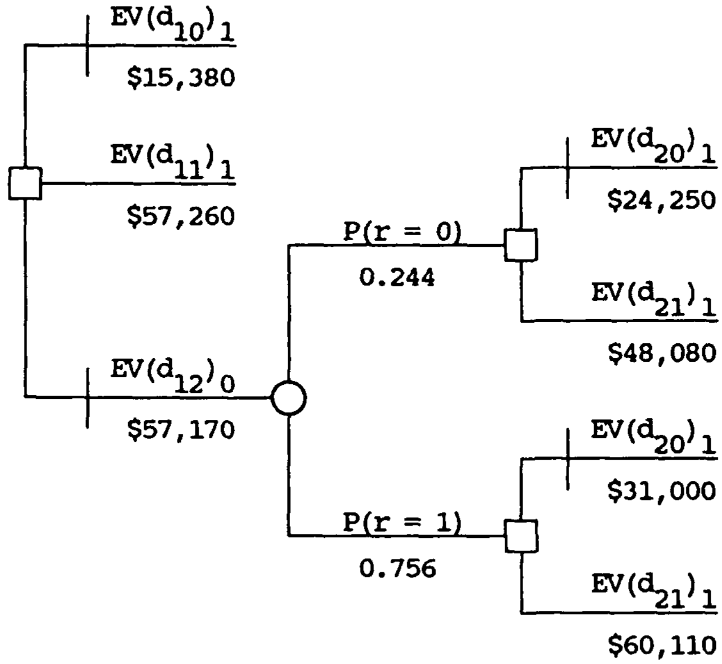


Figure 5. Extensive form analysis at the end of year one. Revisions are made after obtaining year one sample information, and applying values for r_e and n_e with r' and n'.

become the best option through the addition of our equivalent sample size concept to conventional Bayesian methods.

CONCLUSIONS

In this paper, we have developed concepts for continuous models that have natural conjugate distribution families. They have been developed within an equipment replacement/capacity expansion framework, and the key ideas were:

(1) demonstrating how Bayesian techniques can be applied to equipment replacement/capacity expansion problems.

(2) demonstrating how a beta prior distribution can be readily fit to 3-point PERT-type estimates, and how these estimates can be refined through the use of the incomplete beta function (examining probability values for specific intervals of interest) and the equivalent sample size concept (that reflects our initial beliefs in the quality of the estimates).

(3) applying the concept of equivalent sample size to the observed results (a comparison of estimate quality and result typicality), in beta distribution revisions.

(4) the illustration of these techniques in a case study presentation of an actual decision problem.

In the case study we used the beta conjugate family, and obtained a prior distribution candidate solution by simultaneously solving PERT and beta mean and variance equations. We refined that prior belief by keying on its probability of negative return, a measure of effectiveness that highlights the best performing alternative in an

incremental analysis. When we revised that prior belief, we used sample information that incorporated the equivalent sample size concept. The result we obtained was not possible under conventional Bayesian methods, and it led us to a new decision strategy that had a greater expected value than our initial (time 0) strategy.

The new strategy demonstrates that incorporation of these procedures can lead to more efficient use of capital, and greater return on investment. However, a critical consideration to applying these methods is that the user have a thorough understanding of the quantity and quality of both the initial and sample information. The assignment of "weight" factors must reflect the relative strengths of the information, as misapplication of these concepts will give distorted information.

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APPENDIX A. Beta distributions fit to 3-point PERT-type approximations.

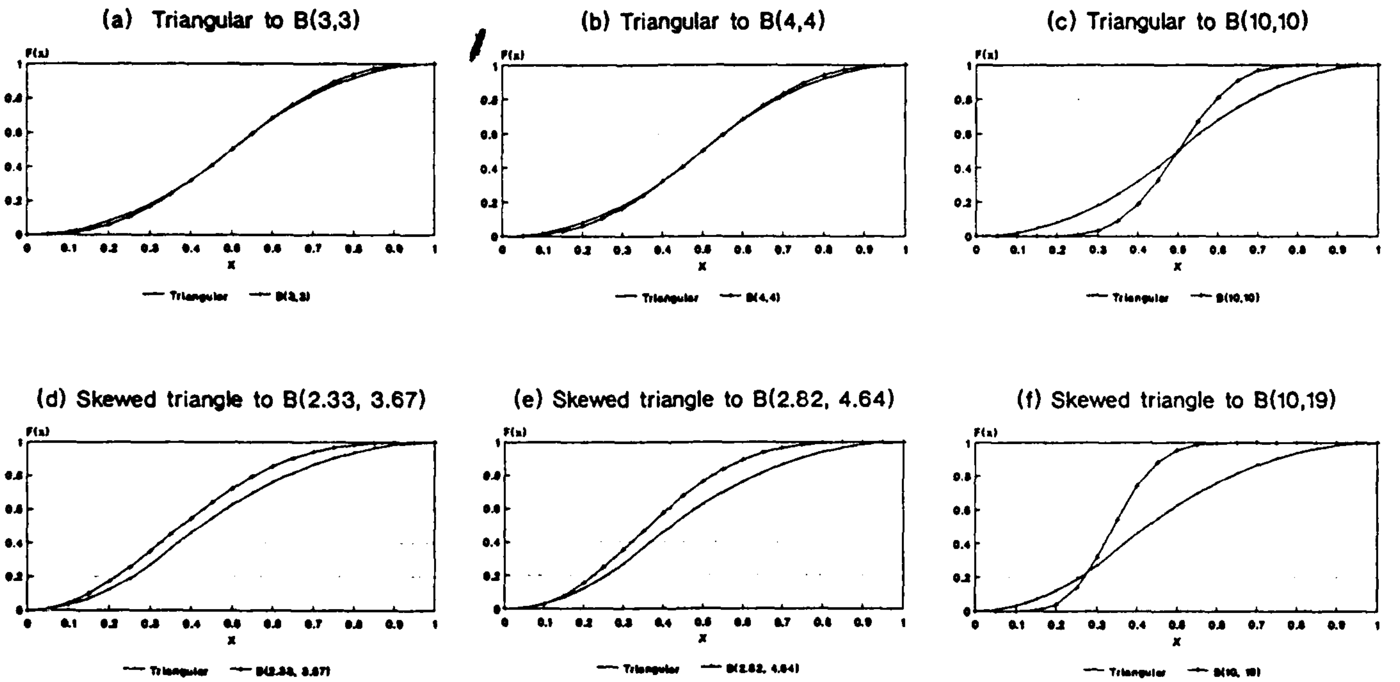
Many capital budgeting proposals, including equipment replacement projects, are prepared as 3-point approximations of the anticipated performance (pessimistic, most likely, and optimistic). The uncertainty described these estimates has frequently been addressed by probabilistic representation as triangular distributions. This is especially true in simulation studies. We feel that this uncertainty can be modeled equally well by variations of the beta distribution (which has other advantageous properties). An earlier work [Keefer and Bodily (1983)] examined several methods that could approximate a distribution's mean and variance, based on 3-point estimates of performance, and tested those approximations with a series of beta distributions. As an extension of some of the concepts presented there, we will examine the similarities of the cumulative distribution functions (c.d.f.'s) of certain triangular and beta distributions.

When the beta distribution shape parameters (α, β) are relatively small, the beta and triangular c.d.f.'s are nearly identical. In fact, the slope of a right triangle (with the 90° angle on the left) is identical to a beta distribution with (α, β) = (1, 2). To demonstrate the similarities of the c.d.f.'s, we will use two different values of the distribution mode (for a 0 to 1 range of x): (1) the mode is 0.5, and (2) the mode is 1/3. In the symmetric case, we compare the triangular c.d.f. to beta distribution c.d.f.'s for (α, B) sets of (3, 3), (4, 4), and (10, 10). The set (3, 3) is used because it is a development from the linear solution of the PERT approximation of the

mean, equation (8), and the beta distribution equation (1) for the mean. The set (4, 4) is used because it is the solution to the PERT and beta equations when the assumption that the Var(x) = $(1 / 6)^2$ is included. The set (10, 10) is used because it represents a distribution where the prior beliefs are more strongly held than the first two cases. As seen in Figure A-1, parts (a) and (b) show that there is little difference between the c.d.f.'s for the triangular distribution and the B(3, 3) and B(4, 4) distributions, respectively. However, the c.d.f. for the B(10, 10) distribution shows a much greater difference.

The second situation, using the skewed distribution with x_m = 1/3, examines the beta parameter sets (2.333, 3.667), (2.82, 4.64), and (10, 19). The set (2.333, 3.667) corresponds to the previously mentioned linear solution, the set (2.82, 4.64) incorporates the variance assumption, and the set (10, 19) represents the stronger belief example. As shown by items (d) and (e), the beta distributions described by the sets (2.333, 3.667) and (2.82, 4.64) closely approximate the triangular distribution's c.d.f. Item (f) shows that there is considerable difference between the B(10, 19) and the triangular distribution.

As shown by the figures, when the α and β values are low, as when they are within the range of the PERT and beta equation simultaneous solutions (either with or without the variance assumption), the triangular and beta distributions are nearly identical. (For brevity, cases with greater asymmetry were not presented. However, those cases show that the more pronounced the asymmetry, the fit is increasingly



Note: Items (a) through (c) are for symmetric distributions.
Items (d) through (f) are for skewed distributions, with mode at 0.333.

Figure A.1. Comparative cumulative distribution functions for select triangular and beta distributions.

poorer for increasing values in the (α, β) parameter sets.) Thereby, since the triangular distribution was an acknowledged approximation of prior beliefs, using the beta distribution will not present a radical change in those beliefs. In fact, when the estimates represent strongly held prior beliefs, a properly fitted beta distribution will provide a much more accurate representation than the triangular distribution. Lastly, modeling the prior beliefs with the beta distribution will allow us to use a natural conjugate family of distributions.

APPENDIX B. BASIC computer program used to obtain initial and refined beta distribution shape parameters.

The attached BASIC computer program takes the initial three-point PERT-type estimates and uses the cubic equation to determine an initial candidate solution for the beta distribution shape parameters, α and β. The user can implement those parameters, or refine them, as desired. The program can then be used to find probability values for specific intervals, and this information is provided as feedback to the user in the parameter refinement process. The program uses Simpson's approximation technique to find the probability values of the Incomplete Beta Function. The program also provides the user with summary information on the evaluated beta distribution.

```
10 DIM ROOT(3)
20 '  This program is designed to find beta distribution shape
30 '  parameters from a given 3-point estimate.  The estimate may
40 '  be standardized (0 to 1) or nonstandard values.
50 '
60 INPUT " Enter the most optimistic estimate ";HIGH
70 INPUT " Enter the most likely estimate     ";MOSTL
80 INPUT " Enter the most pessimistic estimate";LOW
90 '
100 '  From these values, a standardized mode is computed for use in
110 '  the determination of Beta distribution shape parameters.  This
120 '  program will handle nonstandard values, and it also assumes
130 '  that the optimistic and pessimistic values are true end points.
140 '
150 A=LOW  : B=HIGH
160 XM= (MOSTL - LOW)/(HIGH - LOW)
170 '
180 '  with this value, the beta shape parameters, alpha and beta, can
190 '  be found by solving the cubic equation that results from the
200 '  simultaneous equations for the distribution mode and variance,
210 '  assuming that the variance = 1/36
220 '
230 '  to solve the cubic equation, let
240 '
250 P=(7.*XM) - (36.*XM^2.) + (36.*XM^3.)
260 Q=-20.*XM^2.
270 R=-24.*XM^3.
280 '
290 '  use P, Q, and R to find the branching value D
300 '
310 Y1=Q - ((P^2.)/3.)
320 Y2=((2./27.)*(P^3.)) - (P*Q/3.) + R
330 D=((Y2^2.)/4.) + ((Y1^3.)/27.)
340 '
350 '  branch on D if it is <, =, > 0
360 '
370 IF (D < 0) THEN GOTO 410
380 IF (D = 0) THEN GOTO 670
390 IF (D > 0) THEN GOTO 900
400 '
410 '  determine the real cubic roots when D < 0, (three real roots)
420 '
430 '  find the sign of COS(PHI), then angle PHI
440 '
450 IF (Y2 < 0) THEN CSPHI=(((Y2^2.)/4.)/(-1.*((Y1^3.)/27/)))^.5
                 ELSE CSPHI=-1.*(((Y2^2.)/4.)/(-1.*((Y1^3.)/27/)))^.5
460 DEF FNARCCOS(X)=1.570796327-ATN(X/SQR(1-X*X))
470 PHI=FNARCCOS(CSPHI)
480 '
490 '  now find the three roots
500 '
510 FOR I=0 TO 2
```

```
520 ROOT(I + 1)=(2.*(SQR(-1.*Y1/3.)))*COS(PHI/3. + (2.*3.141592/3.)*I)
                        - (P/3.)
530 NEXT I
540 '
550 '  with these roots, find the corresponding alpha and beta values
560 '
570 FOR I=1 TO 3
580 ALPHA=ROOT(I) + 1.
590 BETA=((ALPHA - 1.)/XM) - ALPHA + 2.
600 PRINT " Candidate values for Alpha and Beta are ";ALPHA;BETA
610 NEXT I
620 '
630 '  then go to the incomplete beta function
640 '
650 GOTO 1320
660 '
670 '  Determine the cubic roots when D = 0 (3 roots, at least 2 =)
680 '
690 '  branch on the sign of Y2
700 '
710 IF (Y2 < 0) THEN GOTO 750
720 ROOT(1)=(-2.*(-1.*Y1/3.)^.5) - (P/3.)
730 ROOT(2)=((-1.*Y1/3.)^.5) - (P/3.)
740 GOTO 780
750 ROOT(1)=(2.*(-1.*Y1/3.)^.5) - (P/3.)
760 ROOT(2)=(-1.*(-1.*Y1/3.)^.5) - (P/3.)
770 '
780 '  with these roots, find the alpha and beta candidates
790 '
800 FOR I=1 TO 2
810 ALPHA=ROOT(I) + 1.
820 BETA=((ALPHA - 1.)/XM) - ALPHA + 2.
830 PRINT " Candidate values for Alpha and Beta are ";ALPHA;BETA
840 NEXT I
850 '
860 '  go to the incomplete beta function
870 '
880 GOTO 1320
890 '
900 '  Determine the single real root when D > 0, branching on the
910 '      value of Y1.
920 '
930 IF (Y1 < 0) THEN GOTO 1190
940 '
950 '  for D > 0, and Y1 > 0, find angle PSI
960 '
970 IF (Y2 < 0) THEN CTN2PSI=(((Y2^2.)/4.)/((Y1^3.)/27.))^.5
                ELSE CTN2PSI=-1.*(((Y2^2.)/4.)/((Y1^3.)/27.))^.5
980 DEF FNARCCOT(X)=1.570796327 - ATN(X)
990 PSI=(FNARCCOT(CTN2PSI))/2.
1000 '
1010 '  then find angle PHI
```

```
1020 '
1030 TNPHI=(TAN(PSI))^(1./3.)
1040 PHI=ATN(TNPHI)
1050 '
1060 '  now, find the real root
1070 '
1080 DEF FNCOT(X)=1./TAN(X)
1090 TERM=(2.(SQR(Y1/3.))*FNCOT(2.*PHI)) - (P/3.)
1100 '
1110 '  next, find the values for Alpha and Beta, and go to the
1120 '  incomplete beta function steps
1130 '
1140 ALPHA=TERM + 1.
1150 BETA=((ALPHA - 1.)/XM) - ALPHA + 2.
1160 PRINT " Candidate values for Alpha and Beta are ";ALPHA;BETA
1170 GOTO 1320
1180 '
1190 '  Determine the real root when D > 0 and Y1 < 0.
1200 '
1210 BIGA=((-1.*Y2/2.) + (D^.5))^(1./3.)
1220 BIGB=((-1.*Y2/2.) - (D^.5))^(1./3.)
1230 TERM=BIGA + BIGB
1240 '
1250 '  next, find the values for Alpha and Beta, and go to the
1260 '  incomplete beta function steps
1270 '
1280 ALPHA=TERM + 1.
1290 BETA=((ALPHA - 1.)/XM) - ALPHA + 2.
1300 PRINT " Candidate values for Alpha and Beta are ";ALPHA;BETA
1310 '
1320 '  This part of the program uses the derived parameters to find
1330 '  intervals of the beta cdf, by the incomplete beta function.
1340 '  The beta cdf is approximated by Simpson's rule.  The user has
1350 '  just seen the candidate Alpha and Beta values, and he must
1360 '  make his parameter value selections.
1370 '
1380 INPUT " Enter the desired value for ALPHA ";ALPHA
1390 INPUT " Enter the desired value for BETA  ";BETA
1400 ALPHA=ALPHA - 1.
1410 BETA=BETA - 1.
1420 '
1430 '  When computing the incomplete beta function, the constant
1440 '  inside the integral can be factored and canceled (as it appears
1450 '  in both numerator and denominator.  This leaves the function
1460 '  inside the integral as
1470 '
1480 DEF FNF(X)=(X - LOW)^ALPHA * (HIGH - LOW)^BETA
1490 '
1500 '  first, compute the denominator value
1510 '
1520 GOSUB 1900
1530 CONSTANT=S
```

```
1540 '
1550 '  We can now compute the probability over any desired interval
1560 '  within the range established by the optimistic and pessimistic
1570 '  values.  For initial information to the user, the cumulative
1580 '  probability from the pessimistic value to the mode is found.
1590 '
1600 B=MOSTL
1610 GOSUB 1900
1620 BPROB=S/CONSTANT
1630 PRINT " "
1640 PRINT " For the interval from ";A;" to ";B;", the cumulative"
1650 PRINT " probabilities for your BETA distribution and a "
1660 PRINT " triangular distribution using the 3-point approximation"
1670 PRINT " are, Beta distribution probability ";BPROB
1680 PRINT " Triangular distribution probability";XM
1690 PRINT " "
1700 PRINT " Enter codes for next investigation -"
1710 PRINT "    1 = other intervals, with current Alpha and Beta"
1720 PRINT "    2 = change Alpha, Beta, and Interval"
1730 PRINT "    3 = summarize distribution information"
1740 PRINT "    4 = Quit"
1750 INPUT MORE%
1755 IF (MORE% = 1) THEN GOTO 1800
1760 IF (MORE% = 2) THEN GOTO 2200
1770 IF (MORE% = 3) THEN GOTO 2400
1780 IF (MORE% = 4) THEN GOTO 1870 ELSE GOTO 1700
1790 PRINT " "
1800 INPUT " Enter the upper limit for the interval of interest";B
1810 INPUT " Enter the lower limit for the interval of interest";A
1820 PRINT " "
1830 GOSUB 1900
1840 PRINT " Probability for interval ";A;" to ";B;" is ";S/CONSTANT
1850 PRINT " "
1860 GOTO 1700
1870 END
1900 '  This subroutine computes the area for the interval selected,
1910 '  by using Simpson's rule (without error term)
1920 '
1930 H=B - A
1940 T=(FNF(A) + FNF(B)) * H/2.
1950 GOTO 1970
1960 T=(T + M)/2.
1970 M=0
1980 FOR X=A + H/2. TO B STEP H
1990 M=M + FNF(X)
2000 NEXT X
2010 M=M * H
2020 S=(T + 2.*M) / 3.
2030 H=H/2.
2040 '
2050 '  The stopping rule uses a comparison of the average f(x) values
2060 '  for the interval endpoints and subinterval midpoints.  This
```

```
2070 '  approximation's beta probabilities were compared to a sample
2080 '  of distributions, using the same Alpha and Beta parameters,
2090 '  from Pearson's tables of the incomplete beta function, and
2095 '  were accurate to four decimal places, or better
2100 '
2110 IF ((ABS(T - M) / ABS(S)) > .001) THEN 1960
2120 RETURN
2200 '
2210 '  This subroutine computes probabilities for new shape parameters
2220 '
2230 INPUT " Enter value for ALPHA ";ALPHA
2240 INPUT " Enter value for BETA  ";BETA
2250 INPUT " Enter range lower endpoint ";LOW
2260 INPUT " Enter range upper endpoint ";HIGH
2270 ALPHA=ALPHA - 1.
2280 BETA=BETA - 1.
2290 A=LOW  :  B=HIGH
2300 GOSUB 1900
2310 CONSTANT=S
2320 INPUT " What is the lower limit for the interval of interest ";A
2330 INPUT " What is the upper limit for the interval of interest ";B
2340 GOSUB 1900
2350 PRINT " "
2360 PRINT " Probability for interval ";A;" to ";B;" is ";S/CONSTANT
2370 PRINT " "
2380 GOTO 1700
2390 RETURN
2400 '
2410 '  This subroutine summarizes the current distribution information
2420 '
2430 ALPHA=ALPHA + 1.
2440 BETA=BETA + 1.
2450 BMEAN=ALPHA / (ALPHA + BETA)
2460 BMODE=(ALPHA - 1.) / (ALPHA + BETA - 2.)
2470 BVAR=(ALPHA*BETA) / (((ALPHA + BETA)^2.) * (ALPHA + BETA + 1.))
2480 PRINT " "
2490 PRINT " The Beta distribution you have selected has"
2500 PRINT "    Interval ";LOW;" to ";HIGH
2510 PRINT "    Alpha    ";ALPHA
2520 PRINT "    Beta     ";BETA
2530 PRINT "    Mean     ";BMEAN*(HIGH - LOW) + LOW
2540 PRINT "    Std Dev  ";(BVAR^.5)*(HIGH - LOW)
2550 PRINT "    Mode     ";BMODE*(HIGH - LOW) + LOW
2560 GOTO 1700
2570 RETURN
```

<u>APPENDIX C</u>. Model development and case study application for a normal prior distribution.

<u>MODEL DEVELOPMENT</u>

When the 3-point estimates are symmetric, we can model the uncertainty about the cash flow's expected value with a normal distribution. We can use the 3-point estimates to find $E(\mu)$ and $Var(\mu)$. The apparent drawback to using this model is that the 3-point estimate provides no information about the variance of the cash flow's parent population, σ^2. Further, because the challenger processes involve technical innovations, there will also generally be no other historical records to provide input. While we are truly facing an unknown population variance condition, the 3-point estimates give us no information to establish values for $E(\sigma^2)$ and $Var(\sigma^2)$. Fortunately, other circumstances in this problem will permit us to overcome the absence of $E(\sigma^2)$ and $Var(\sigma^2)$. As shown in an earlier work [Prueitt and Park (1989)], uncertainty resolution occurs very slowly in the population variance ($Var(\sigma^2)$ goes to zero slowly), requiring many sample observations. Because many replacement projects have limited sample sizes, they would not realize the effects of that resolution. Thereby, if we develop a value for σ^2, and treat it as a known value, there would be little difference in the end results. As we will show, subsequently, we can use the 3-point estimates with the equivalent sample size concept to develop a value for σ^2. We will then follow conventional procedures for an unknown mean, known variance condition.

When we use the normal distribution, we also follow the belief that the sample information is normally distributed. The information

provided by its likelihood function is generally more descriptive than the Bernoulli process, but we must consider the following:

a. Normality assumptions. The normal distribution models are based on the mean performances of some x_j, $j=1,2,\cdots,n$, that are assumed to be independent identically distributed random variables (IIDRV), with a common mean and variance. The accuracy of this model is adversely affected by undetected trend or seasonal effects.

b. Sample size requirements. The closer the distribution of x_j's is to the normal curve (specifically, the closer the distribution is to being symmetric, or having smaller third central moment, or skewness, values), the smaller n can be, and still provide good accuracy.

c. Period of observation. If we model a project's net present value (including investment as well as return) as the x_j's, then the required observation period is one unit's economic life. This is frequently several years for manufacturing equipment, which makes this an unattractive approach for project net present value. However, we can resolve performance uncertainty if the x_j's occur in "steady-state" conditions, as with periodic cash flows from an on-line process operating at a relatively fixed level of production. Here, the x_j's would be the periodic (monthly, quarterly, yearly, etc.) cash flows, and their uncertainty resolution will be used to refine the project performance estimates, in anticipation of the sequential decision reviews.

Initial prior belief. When we begin fitting a normal distribution to a 3-point estimate, we must consider how to handle the estimate extreme points, realizing that there is only a limited amount of flexibility for any interval probabilities. For this normal model, the optimistic and pessimistic estimates will not represent absolute limits. This gives some flexibility to the estimating process, but, if those estimates are truly absolute limits, then tail probabilities outside of extremes are misrepresented. To fit a normal distribution to a 3-point estimate, the optimistic and pessimistic values must be associated with specific percentiles of the distribution, and the outcome probability for an interval of interest can only be slightly modified. For example, the 3-point estimate of $50, $100, and $150 is modeled as a normal distribution. The outcome probability for each $10 interval from $100 to $150, with the pessimistic and optimistic estimates being used as the (1st, 99th), (5th, 95th), and (10th, 90th) percentiles is listed in Table C.1. It is readily seen that there are minimal differences between the (5th, 95th) and the (10th, 90th) percentile sets, the percentiles most often used with PERT-type equations. The percentiles will be used with the normal c.d.f., to find the standard deviation of the cash flow expected value. For example,

$$x(0.95) - x(0.05) = 2 * 1.6449 * \sigma_\mu$$

and,

$$\sigma_\mu{}^2 = \mathrm{Var}(\mu).$$

Table C.1. Effects of changing percentiles associated with the 3-point estimate extreme values, using the estimate ($50, $100, $150) and a normal distribution model.

	Estimate percentiles and values		
Probability of interest	x(0.01) = $50 x(0.99) = 150	x(0.05) = $50 x(0.95) = 150	x(0.10) = $50 x(0.90) = 100
$P(100 \leq x \leq 110)$	0.179	0.129	0.101
$P(110 \leq x \leq 120)$	0.145	0.116	0.095
$P(120 \leq x \leq 130)$	0.095	0.094	0.083
$P(130 \leq x \leq 140)$	0.050	0.068	0.068
$P(140 \leq x \leq 150)$	0.021	0.044	0.053

The other key parameter in this approach is the parent population standard deviation, σ. If we consider the established relationship [Winkler (1972)]:

$$n' = \sigma^2 / \sigma_\mu^2$$

where n' is the relative weight, or quality, of information for Var(μ) compared to σ^2. We can rearrange the equation such that

$$n' \sigma_\mu^2 = \sigma^2$$

For example, if we were to believe that the estimates are as good as the reports from six periodic cash flows, then n' = 6. By defining σ_μ^2 and n', we obtain a value for σ^2, and we have enough information to use the conventional procedures for a normal distribution in a known variance condition.

<u>Conventional Bayesian procedures for normal distributions</u>. Briefly, when the cash flow variance is assumed known, the uncertainty of the cash flow mean, μ, can be represented as a normal prior density function, of the form:

$$f'(\mu) = (2\pi\sigma'^2)^{-\frac{1}{2}} \exp(-(\mu - m')^2 / 2\sigma'^2)$$

where,

μ is the unknown mean,

m' is the prior estimate of the expected cash flow, E(μ), and

σ'^2 is the prior estimate of the variance of the expected cash flow.

When a sample y, of n observations and observed mean m, is taken from a normally distributed population, the likelihood function combines with the prior belief to form a normally distributed posterior belief:

$$f''(\mu \mid y) = (2\pi\sigma''^2)^{-\frac{1}{2}} \exp(-(\mu - m'')^2 / 2\sigma''^2)$$

The resulting population cash flow distribution is described as $\approx N(m'', \sigma^2)$, and the posterior belief about the expected value of the cash flow is described as $\approx N(m'', \sigma''^2)$. We find m'' and σ''^2 from:

$$\frac{1}{\sigma''^2} = \frac{1}{\sigma'^2} + \frac{n}{\sigma^2} \tag{C-1}$$

$$m'' = \frac{\dfrac{m'}{\sigma'^2} + \dfrac{nm}{\sigma^2}}{\dfrac{1}{\sigma'^2} + \dfrac{n}{\sigma^2}} \tag{C-2}$$

Now, defining the terms

$$n' = \sigma^2 / \sigma'^2 , \qquad \text{and} \qquad n'' = \sigma^2 / \sigma''^2$$

then equation (C-1) can be rewritten as

$$\frac{n''}{\sigma^2} = \frac{n'}{\sigma^2} + \frac{n}{\sigma^2} \tag{C-3}$$

or,

$$n'' = n' + n \tag{C-4}$$

where n' refers to the amount, or relative weight, of information for $\sigma'^2 = \text{Var}(\mu)$, as compared to σ^2. We then use equation C-3 to solve equation C-2:

$$m'' = \frac{n'm' + nm}{n' + n} \tag{C-5}$$

Equivalent sample size concept for the normal distributions. The prior estimate of the cash flow expected value, m', was developed from forecasts for specific economic conditions. As mentioned previously, the conditions that exist during the observation period may not match those in the predictions. If the actual performances vary markedly from the predictions, they may be felt to be more representative of the true performance than the initial estimates. In either case, the

value, or "weight", of the sample information needs adjustment, as compared to the initial information. We make this adjustment by using an equivalent sample size, n_e. We rewrite equations C-3 and C-5 to incorporate the strength of belief in the prior estimate concept (through the inclusion of n'), and to incorporate the strength of belief in the replicability of the sample (through the inclusion of n_e), which gives:

$$\frac{1}{\sigma''^2} = \frac{n'}{\sigma^2} + \frac{n_e}{\sigma^2} \tag{C-6}$$

$$m'' = \frac{n'm' + n_e m}{n' + n_e} \tag{C-7}$$

If the sample information is more accurate than the predictions, then we make $n_e > n$, and σ''^2 will go to 0 faster. If the results are anomalous, then we make $n_e < n$, and σ''^2 will move to 0 more slowly.

As a final note to this section, some authors have stated that when the variance is not known, the sample variance, $s^2 = (\Sigma(x_j - m)^2) / (n - 1)$, can be substituted for the population variance. However, for initial decision formulation, there is no sample variance. After project initiation, this approximation is only appropriate when the number of observations is "sufficiently" large, and use of the sample variance modeling may not be appropriate in many equipment situations, because they only have a small number of replicable elements.

CASE STUDY: APPLICATION OF NORMAL DISTRIBUTION MODEL

The normal model is used when the estimates are symmetrically distributed. Since the case study involves an asymmetric situation, we

will modify it for illustrative purposes. We will assume that the estimates are symmetric, that the newly acquired equipment comes in assemblies that are readily installed, and that production conditions will be stable in the first year. We will assume that the cash flows of years 5 through 8 begin in year 1, and a "steady-state" condition exists. We summarize the modifications by restating the annual cash flow estimates as:

$$x(0.05) = \$58{,}500$$

$$x_m = \quad 63{,}375$$

$$x(0.95) = \quad 68{,}250$$

<u>Determination of prior belief</u>. We will first use the estimates to find the cash flow's expected value, $E(\mu)$, and that expected value's variance, $Var(\mu) = \sigma_\mu{}^2 = \sigma'^2$. This variance must not be confused with the variance of annual cash flows, σ^2. We will treat the estimates as the 5th, 50th, and 95th percentiles of the expected values distribution, such that

$$E(\mu) = m' = \$63{,}375$$

$$Var(\mu) = \sigma'^2 = [(x(0.95) - x(0.05)) \ / \ (2 * 1.6449)]^2 = 2{,}964^2$$

We feel that the expected value estimate may be somewhat inaccurate, due to the technical innovations in the process, and the estimates are comparable to a sample of four cash flow observations. Thereby,

$$n' = 4$$

$$\sigma^2 = n' * \sigma'^2 = 5{,}928^2$$

and σ^2 will be treated as being known with certainty.

Conventional Bayesian methods for remaining time 0 analysis. If the prior belief in μ is normally distributed, with known σ^2, the predictive distribution for the sample mean, m, is also normal, with:

$$E(m) = m' = \$63,375$$

$$Var(m) = \sigma^2 \left(\frac{1}{n'} + \frac{1}{n} \right) = 2,651^2$$

Because the expected outcome, E(m), is equal to the prior belief, there are no changes in the different alternatives' payoffs, and the result keyed decision process of the beta-binomial distribution is not applicable. The information provided by the predictive distribution is the relative reduction in the variance provided by the sample size, n.

Applying the equivalent sample size to the sample data. One year of sample data was recorded, and the incremental cash flow was $67,188. The process was found to operate faster than predicted, creating a larger than anticipated reduction in WIP. There were no anomalous external economic conditions during the year. After a review, it was felt that the sample information had twice as much weight as the initial forecasts. Therefore, when we make the revisions, we use equations C-6 and C-7, with $n_e = 8$, rather than $n = 1$. This gives us:

$$m'' = \frac{4(63375) + 8(67188)}{4 + 8} = \$65,917 = E(\mu)$$

$$\sigma''^2 = \frac{5928^2}{4 + 8} = 1,711^2 = Var(\mu)$$

the posterior Var(μ), when compared to its prior value, reflects a marked uncertainty reduction. If we desire, we can use this information to make predictive intervals about the estimated mean value, but more importantly, we can update the decision alternative payoffs. The posterior mean is then used to revise the expected net

present values for each of the remaining unit initiation times. This leads to a subsequent recomputation of decision alternative payoffs, and once these values are obtained, the firm makes its sequential decision review.

CONCLUSION

Here, we have demonstrated how the information in a 3-point estimate could be used to approximate the values needed in an unknown mean, known variance situation. The population variance was estimated by an application of the equivalent sample size concept, based on a quality of information assessment. The revision of those initial beliefs was also illustrated, using the equivalent sample size concept.

V. DISCRETE APPROXIMATIONS TO CONTINUOUS MODELS IN SEQUENTIAL EQUIPMENT REPLACEMENT DECISIONS

ABSTRACT

Capital budgeting models have been extensively developed for project selection, but little attention has been given to methods for project control through the use of post-audit information for implemented projects. This paper develops an approximation technique that uses post-audit information to resolve the uncertainty about the expected net present value of a technically innovative process. We take a project's three-point, PERT-type estimates of expected cash flow performance and use them to develop a discrete probability distribution that represents our prior beliefs. We then show how the Dirichlet distribution can be used to formulate weighted initial beliefs, and how it can be used with an equivalent sample size concept that results in a flexible and efficient means to revise our initial beliefs. We exercise these concepts in a case study analysis of an actual decision problem.

INTRODUCTION

Manufacturing equipment replacement decisions have not, historically, received as much attention as other capital budgeting decisions (such as initial investments, portfolio composition, etc.).

Past replacement decisions were typified by a current, or defender asset's performance being compared to an alternative, or challenger asset's performance. The performance realizations generally had only small variations from their estimates. In fact, surveys reported that the estimates used in equipment replacement decisions were more accurate than any other investment estimates [Pohlman, Santiago, and Markel (1988)]. Consequently, some firms began to use deterministic methods in these decisions, concentrating on the timing of equipment changeovers [Bean, Lohmann, and Smith (1985)]. This approach effectively ignored the uncertainty in the cash flow estimates.

Currently, technological advances have changed the scope of the replacement decision. Many corporations must now consider state-of-the-art processes that include computers, robotics, flexible manufacturing cells, and just-in-time manufacturing. Many of these new alternatives require the restructuring of entire production processes, essentially expanding the replacement decision from machine for machine, to process for process considerations. A frequently occurring example is the comparison of factory machine shop layouts and flexible manufacturing cell organizations. The drawback in this expanded decision is that while all the technical innovations represent great potential, they do not have proven performances, in the form of detailed historical records. Surveys report that when processes vary markedly from previous corporation practices, there tends to be much more variation between performance realizations and their estimates [Cook and Rizzutto (1989)]. Because these broad scope alternatives

have increased variability, corporations can no longer ignore the uncertainty that accompanies these replacement comparisons.

The unknown aspects of the performance estimates can be overcome by the process of uncertainty resolution [Bierman and Hausman (1972)]. When firms use a periodic, or sequential decision review policy, this resolution permits them to make more informed decisions. In this type of decision structure, when the investment alternatives involve items that can be replicated, the uncertainty resolution provided by the sampled portion may even lead to some initially unattractive investments being profitable [Bierman and Rao (1978)].

We will present a method that can be used in a sequential decision making environment, that provides uncertainty resolution to the generalized case (the special case being when the prior belief and sample likelihood function form a natural conjugate family of distributions). This paper is the fourth in a series [Prueitt and Park (1989(a-c))] that explores post-audit information and the replacement problem decision. Here, the generalized approach is to develop discrete approximations to continuous prior beliefs, record the sample results, and place the observations in discrete categories that correspond to those in the prior belief. Combining these discrete distributions in a Bayesian framework relaxes the assumptions used in the continuous case. Specifically, those concerning data sample sizes and that the sample data belong to a particular family of distributions. We will apply this method to a case study of an actual decision problem.

MODEL DEVELOPMENT

When we consider the expected net present value (NPV) of a project's cash flow, three natural factors lead to its being modeled as a continuous random variable: (1) the divisible nature of the data (or dollar count), (2) the uncertainty in the timing of cash flow occurrences, and (3) the process of discounting the cash flow to a particular point in time. However, circumstances may preclude handling the cash flow as a continuous variable, requiring us to use a discrete approximation, instead.

Conditions making discrete approximations appropriate

Some of the conditions that may lead us to using a discrete approximation are:

1. The uncertainty about the cash flows may not be representable by a smooth probability density function (p.d.f.), as: (a) the prior distribution is based on some development of a histogram that does not fit any particular distribution, (b) varying economic environmental conditions may have drastic effects of the cash flows, creating a multiple mode condition, or (3) the cash flow may be developed on an incremental basis (alternative A - alternative B), and that incremental distribution may be irregular, due to different alternative reactions to specific economic conditions.

2. The prior beliefs and the sample likelihood function may <u>not</u> form a natural conjugate family of distributions. For example, the prior distribution may be modeled as a beta distribution, while the data observations are represented as samples from a normal distribution

(earlier works [Winkler (1972) and Jones (1977)] have only presented concepts using a normal likelihood function, without addressing the appropriate steps if the sample distribution is obviously skewed).

3. The amount of sample information to be available may be limited (a continuous variable with a sample size of one or two) to the extent that it is not appropriate to model them as a normal, or any other type of distribution.

These conditions require us to make an adjustment before we can apply Bayesian revision concepts on a generalized basis, and that is to convert the range of continuous outcomes to a set of discrete intervals.

General assumptions

Due to uncertainty, the expected NPV of a given project is treated as having a probability distribution with some unknown parameters. This NPV is the discounted sum of a series of periodic cash flows, and each of these periodic cash flows can likewise be described as having some descriptive probability function (see Figure 1). In this model we are interested in the situation where the periodic cash flows are modeled as identically distributed random variables. This description most often describes the projected cash flows of equipment replacement projects. This includes technically innovative projects, which tend to have tremendous uncertainty over the region of cash flow occurrences, but stable cash flows within the true operating portion of that region. This is attributable to the fact that once installed and operating, these projects exist in stable economic environments. If the number of periodic cash flows is large (greater than 30), the net present value

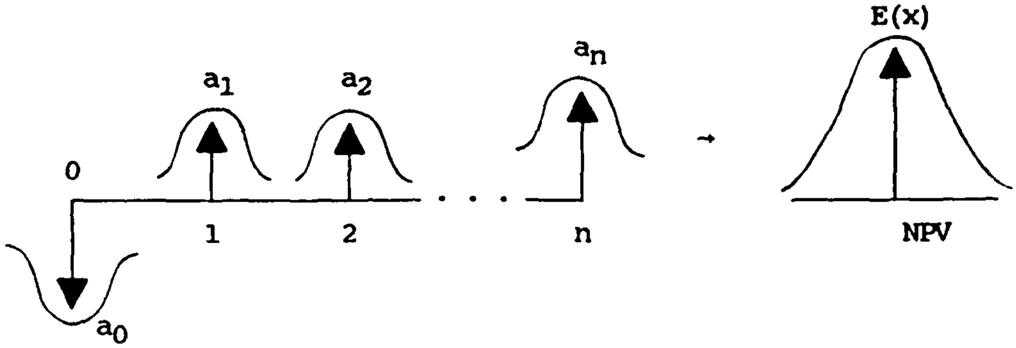


Figure 1. Probabilistic nature of periodic cash flows and the resulting net present value.

of the cash flow observations can be treated as being normally distributed, as per the Central Limit Theorem. However, we are interested in the situations where there are only a small number of observations, and such an elegant property is not reasonably assumable. Therefore, we will instead model the distribution of periodic cash flows with a discrete approximation.

We also assume that the projects of interest have replicable cells, and the firm uses a periodic decision review process. This permits us to record sample results, use that information in Bayesian revision procedures to update our initial beliefs, and make appropriate decision strategy adjustments.

Development of initial periodic cash flow distribution

The capital projects we are interested in are most frequently prepared as three-point, PERT-type estimates (optimistic, most likely, and pessimistic). The 3-point periodic cash flow estimates are discounted at the firm's minimum attractive rate of return (MARR) to provide the project expected NPV, with upper and lower bounds. We will refine this process by keying on the initial 3-point estimates of the periodic cash flows. We propose that the initial range of outcomes described by the 3-point estimates be divided into discrete, non-overlapping intervals. (It is not necessary that the intervals be of equal length, but that approach will make it easier to portray the results.) Once those intervals are designated, we establish corresponding probabilities of occurrence for each interval. An earlier work [Prueitt and Park (1989(c))] describes how the 3-point estimates can be fit to non-standard beta distributions, and how the

incomplete beta function can provide the cumulative probabilities for specific distribution intervals. (We will use this approach in the case study.) Another approach would be to subjectively assign probabilities to the intervals.

These initial interval probabilities, θ_j's, are then examined and refined as appropriate. The revision can be done by graphically comparing the θ_j's, and making adjustments, one interval relative to another, until it accurately describes our beliefs. (This refinement is not necessary if the initial probabilities are subjectively assigned.) This flexibility permits us to describe prior beliefs that could not be adequately illustrated by a pure beta, or other, distribution.

A discrete prior distribution can be used to directly compute a project's expected NPV, through the use of the multinomial probability density function. However, that application would not take advantage of the available sample information. Therefore, we will instead use the θ_j's to develop descriptive parameters, α_j's, for a Dirichlet distribution, which has natural conjugate distribution qualities.

The α_j's are determined by first making a subjective relative assessment of the quality of the initial estimates (both the 3-point and the developed θ_j's) as compared to the amount of sample information that can be collected during one time period, under <u>anticipated</u> economic conditions. We will call this comparison an information quality factor (IQF). For example, if we feel that the estimates are twice as informative as a single periodic cash flow, then the IQF = 2. If the first period will have six cells reporting cash flows, and we

feel that those results will be as informative as our estimates, then the IQF = 6. We then obtain the descriptive α_j's from the relation:

$$\alpha_j = \theta_j * \mathrm{IQF} \qquad \text{for } j=1, \ldots, k$$

With the initial Dirichlet distribution defined, we are prepared for the revision process.

Bayesian revision model

The updating process will use the Dirichlet-multinomial family of natural conjugate distributions. The Dirichlet distribution is sometimes referred to as the multivariate beta distribution, and this method can be thought of as a generalization of the beta-binomial conjugate family. The process uses a Dirichlet prior distribution, samples that form a multinomial likelihood function, and results in a Dirichlet posterior distribution.

In the formulation, it is given that a particular cash flow result will fit into exactly one of k different outcome categories. The probabilities associated with each of the k respective categories comprises the random vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ with ($\theta_j \geq 0$; j=1, ..., k) and $\Sigma\theta_j = 1$. There also exists a parametric shape vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ with ($\alpha_j > 0$; j=1, ..., k), such that the probability density function of $f(\theta \mid \alpha)$ will form a Dirichlet prior distribution

$$f(\theta \mid \alpha) = \frac{\Gamma(\alpha_1 + \ldots + \alpha_k)}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_k)} \theta_1^{\alpha_1 - 1} \cdots \theta_k^{\alpha_k - 1} \qquad (1)$$

where,

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt.$$

The α_j are not limited to integer values, and if k=2, the result is the familiar beta distribution. Then, the observed sample is a random vector $\mathbf{x} = (x_1, \ldots, x_k)$ and $(x_j \geq 0; j=1, \ldots, k)$, where a given x_j represents the number of observations falling into a kth category, with $\Sigma x_j = n$. This observed sample uses the vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$, to form a multinomial likelihood function

$$f(\mathbf{x} \mid n, \boldsymbol{\theta}) = \frac{n!}{x_1! \ldots x_k!} \theta_1^{x_1} \cdots \theta_k^{x_k} \qquad (2)$$

This likelihood function combines with the prior distribution to form a Dirichlet posterior distribution

$$f(\boldsymbol{\theta} \mid \boldsymbol{\alpha}, \mathbf{x}) = \frac{\Gamma(n + \Sigma\alpha_i)}{\Gamma(\alpha_1 + x_1)\ldots\Gamma(\alpha_k + x_k)} \theta_1^{\alpha_1 + x_1 - 1} \cdots \theta_k^{\alpha_k + x_k - 1} \qquad (3)$$

This permits direct computation of the prior and posterior values of $\boldsymbol{\theta}$ (θ_j', and θ_j'', respectively) from the Dirichlet distributions' shape parameters [Johnson (1960), and Johnson and Kotz (1969)]:

$$\theta_j' = \alpha_j / \Sigma\alpha_j \qquad (4)$$

$$\theta_j'' = (\alpha_j + x_j) / (n + \Sigma\alpha_j) \qquad (5)$$

In using this revision model, we must keep in mind that the natures of the α_j's are such that the stronger one believes a parameter is true, the larger its corresponding α_j (as described by the IQF). For example, a given Dirichlet distribution has three equiprobable states of nature ($\theta_1 = \theta_2 = \theta_3 = 1/3$). These probabilities can be described by $\boldsymbol{\alpha} = (100,100,100)$, a strongly held prior belief, or by $\boldsymbol{\alpha} = (1,1,1)$, a weakly held prior belief. Ten units are sampled, with all ten outcomes favoring θ_1 ($n = 10$, $x_1 = 10$, $x_2 = x_3 = 0$). For the parameter set, $\boldsymbol{\alpha} = (100,100,100)$, the posterior values for θ_j become

$$\theta_1'' = \frac{\alpha_1 + x_1}{n + \Sigma\alpha_j} = \frac{100 + 10}{10 + 300} \approx 0.355$$

and,

$$\theta_2'' = \theta_3'' = (100 + 0)/(10 + 300) \approx 0.323.$$

On the other hand, using the second set, $\alpha = (1,1,1)$, yields

$$\theta_1'' = \frac{1 + 10}{10 + 3} \approx 0.846$$

and,

$$\theta_2'' = \theta_3'' = (1 + 0)/(10 + 3) \approx 0.077$$

The second shape parameter set reacts very quickly to the sample observations, while the first set is hardly disturbed.

Another item that should be noted stems from earlier works with this natural conjugate distribution that focused on goodness-of-fit and independence tests [Good (1967)]. These works used symmetric (equiprobable) Dirichlet distributions, with the notation

$$f(\theta \mid \alpha) = \frac{\Gamma(\alpha k)}{\Gamma(\alpha)^k} \theta_1^{\alpha-1} \ldots \theta_k^{\alpha-1} \tag{6}$$

where,

$$\alpha_1 = \alpha_2 = \ldots = \alpha_k = \alpha$$

and the α was referred to as a "flattening" constant. When this equiprobable condition was used in a situation where there was a natural grouping of categories, or overlapping categories for a continuous variable, it was shown that regeneration would eventually lead to loss of the Dirichlet properties [Good (1965)]. However, this problem was overcome by adding a distribution for α (maintaining the α_i's as distinct elements) which leads the posterior distribution away from equiprobability [Lindley (1980), and Good (1983)]. There were no

such problems when an asymmetric prior distribution was used, nor were any problems noted when working in a noncontingency table structure. Here, we will employ the generalized shape parameter $\alpha = (\alpha_1, \ldots, \alpha_k)$ with distinct intervals to avoid any potential problems.

Equivalent sample size concept - for observed results

We stated previously that the predicted sample results are based on anticipated economic environmental conditions. As we collect the cash flow data, we must make assessments of the occurring economic conditions to determine the typicality, or repeatability of the observation. If we have disparity between the assumed and observed economic conditions, we may need to adjust the numbers of the observations, x_j's, with the equivalent sample size concept [Prueitt and Park (1989(c))]. If the estimates included processes that were misjudged, and we feel that the sample is more representative of the true state than the estimates, we will want to increase the observed sample size. If there are unanticipated conditions that are of a once and temporary nature, we will want to reduce the observed sample size. In either case, if we use the equivalent sample size, we must make the adjustments to all categorical results proportionately. As appropriate, we would replace the observations, x_j's, with equivalent observations, x_{e_j}'s, so long as,

$$\frac{x_j}{\Sigma\, x_j} = \frac{x_{e_j}}{\Sigma\, x_{e_j}} \qquad \text{for } j=1, \ldots, k$$

and equation (5) is rewritten as:

$$\theta_j'' = \frac{\alpha_j + x_{e_j}}{\Sigma\alpha_j + \Sigma x_{e_j}} \tag{7}$$

If the economic conditions are as anticipated, then the observed x_j's can be applied directly.

Decision strategy revision

At every decision point, the distribution of periodic cash flows is used to determine the project's expected NPV. As that value changes, the existing decision strategy may need to be revised. Conventional Bayesian methods strive to predict all sample outcomes, and plan decision options for each outcome. However, because we may feel it necessary to apply the equivalent sample size concept (in the handling of anomalous conditions), we cannot predict all possible sample results, as the revision of the periodic cash flows may take unanticipated changes. Therefore, we must review and update our strategy at each periodic decision point.

Method summary

We will briefly summarize the steps for this procedure, which is a generalized approach that can provide an approximation for any prior distribution of beliefs:

1. Develop the prior distribution (incremental or otherwise), considering the range of the periodic cash flow outcomes. Determine the desired number of intervals, and where the partitions should be made. (It is critical that the intervals do not overlap.) The intervals selected will be common to all periods.

2. Determine the initially associated probabilities, θ_j's, for each interval.

3. Determine the values for the shape parameters, α_j's. In making this construction, we must consider the quality of the estimates and the quantity of the sample information and develop the IQF. We then multiply that factor by each of the θ_j's to obtain the respective α_j's.

4. Implement the project, and obtain the sample results.

5. As the data is collected, make assessments of the repeatability, or typicality of the observed results. If there are anomalous conditions, adjust the numbers of observations with the equivalent sample size concept, x_{e_j}.

6. Revise the interval θ_j's, using Bayesian procedures for the Dirichlet distribution, using x_j's or x_{e_j}'s, as appropriate.

7. Use the new θ_j's to determine the new distribution's expected value, and make decision strategy adjustments, as necessary.

8. Continue the process for the life of the project.

To illustrate these procedures, the following section is a case study of an actual decision problem.

CASE STUDY

The following case study of an actual decision problem is provided as an illustration of the techniques we have developed.

General situation

The North American Telecommunications Corporation is a medium sized manufacturer of electronic circuit boards, called "backpanels". They make three series, or "families" of products that are sold to larger computer manufacturing firms. Two of the families are high volume/low marginal return items, and the third is a more complex, customer-tailored product, with a higher marginal return. A critical part of their production process is the alignment of connecting leads, or pins. This task is performed by two machines. One of the devices is an automatic pin insertion machine, or "pin staker", which provides 90% of the pin quantity required on a given backpanel. The remaining 10% is provided by a robotic secondary pinning process. Other steps in the process include preforming, soldering, assembly, inspection, and rework.

The current factory design layout, or defender process, is a machine shop layout, based on operational functions (see Figure 2). This layout was designed to handle large batches of product as they moved through each of the centralized areas. Because each batch requires part specific setups, the batches compete with one another for available work areas. After factory operations began, the orders received were not exactly as the corporation had anticipated. The total order quantities were accurate, but the orders were in more

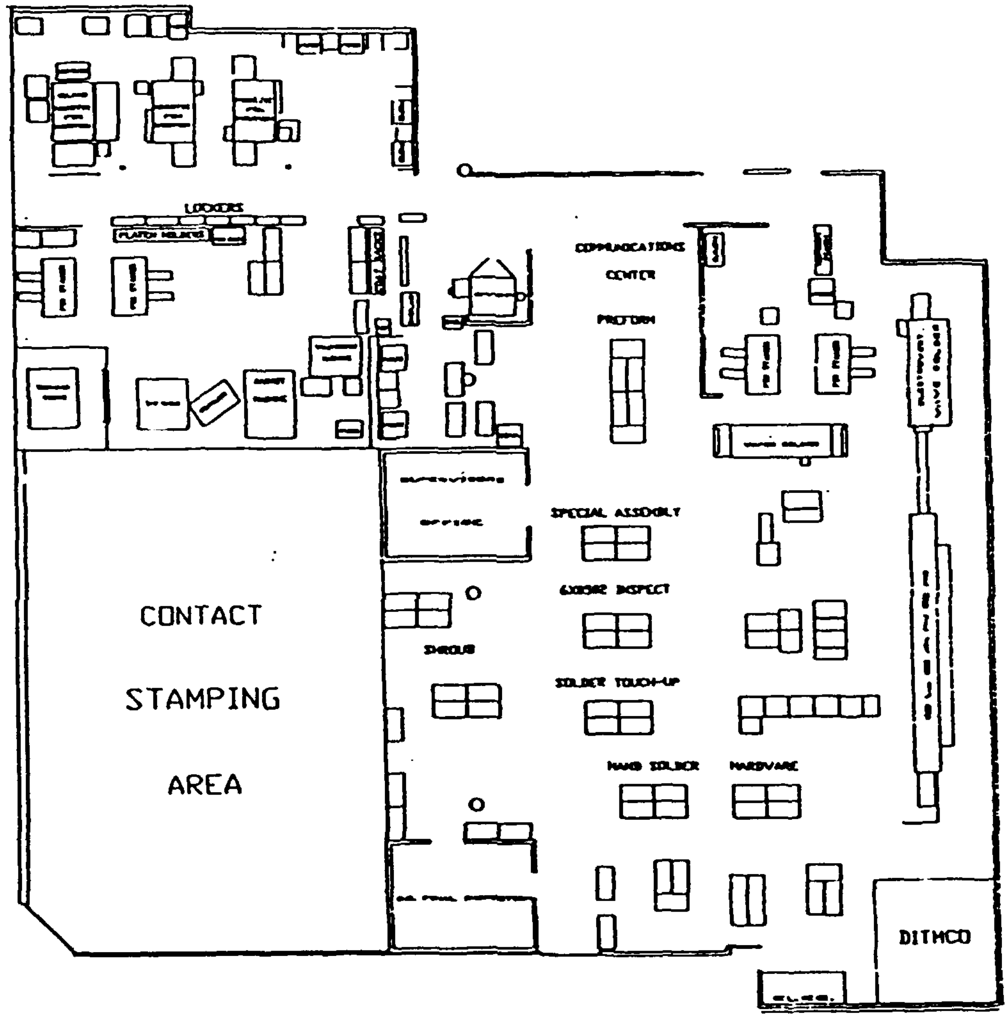


Figure 2. Factory design under existing machine shop organization.

frequent, much smaller lot sizes. This caused plant operations to be characterized by scheduling difficulties, high levels of work in progress (WIP), and only a limited ability to track a specific product through the process. The magnitude of the difficulties is not expected to keep the firm from meeting its projected demands.

To resolve some of the present difficulties, the corporation is considering converting from its present layout to an alternative design, or challenger process. This process will use flexible manufacturing cells and the just-in-time (JIT) manufacturing concept. This alternative will require several additional pieces of equipment, as well as all the existing equipment. The challenger layout is shown in Figure 3. Here, the assembly process is broken into smaller pieces, with specific product families being produced in dedicated cells. In the modified layout, two cells are dedicated (one each) to the high volume families, with the remaining two cells dedicated to the complex products. It is anticipated that this design will ease scheduling and improve the firm's ability to track a specific product.

Economic factors for the immediate, full conversion alternative

The full conversion from the defender to challenger process will require the following expenditures:

Capital equipment (tax depreciable items)-	
Pin staker, installed	$240,591
Robot, installed	86,194
Lubrication stations, 2, @$7,000	14,000
Parts replenishment units, 4, @$625	2,500
Other costs (tax expense items)	76,780
Total	$420,065

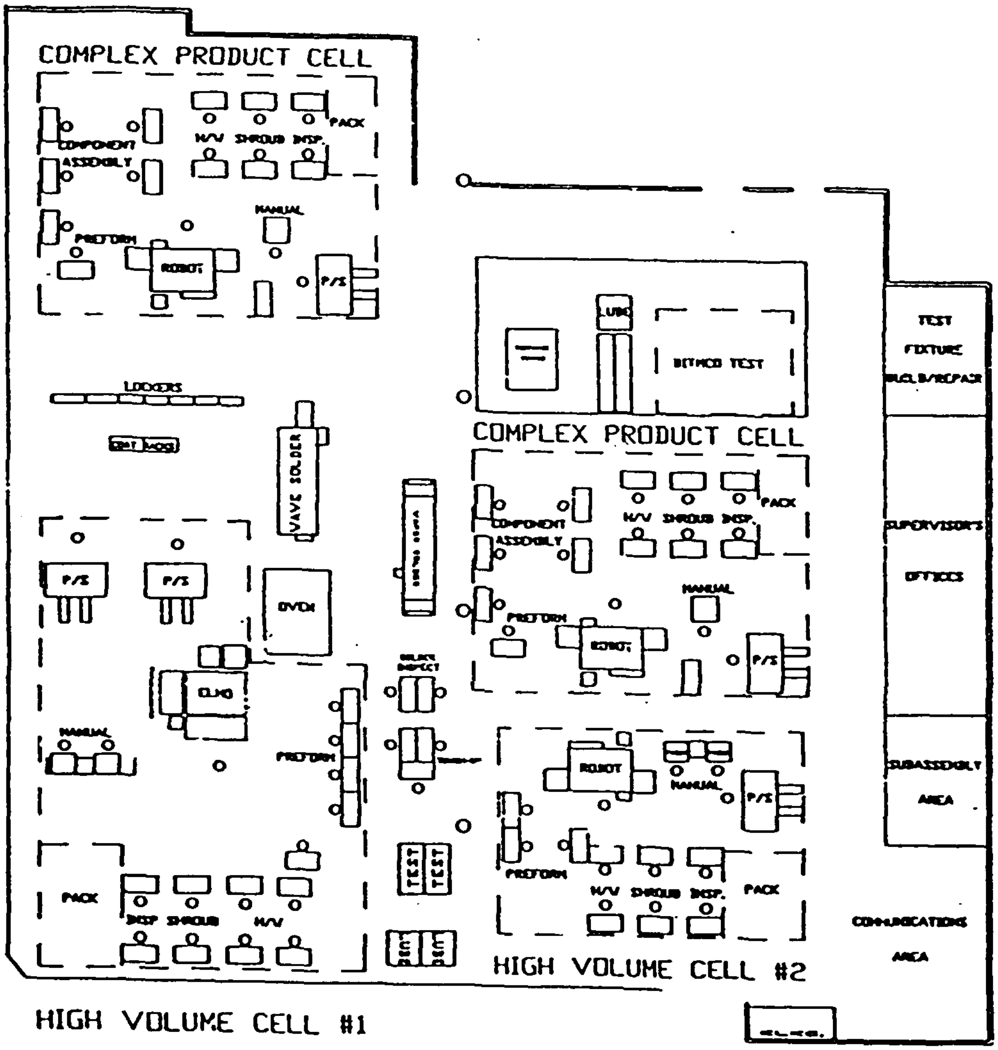


Figure 3. Factory design under proposed flexible manufacturing cell organization.

The capital equipment is classified as a five-year property, and will be depreciated by the Modified Accelerated Cost Recovery System (MACRS). Some of the quantifiable benefits are expected to be from WIP and labor costs. The factory currently carries an average of 2,200 backpanels in WIP, at an average cost of $120 each. The challenger process is expected to reduce WIP by 50%, to 1100 units. This will create a one-time release of $132,000 of working capital back to the firm, and a reduction of non-capital components of the inventory holding costs of $9,240 per year. The holding cost savings are for reductions in taxes, insurance, handling, storage, and item accountability. These costs are rated as being 7% of the inprocess unit cost ($120). Additionally, under the challenger configuration, each backpanel is expected to be completed in 6% less working time. This improved labor efficiency is expected to generate $99,720 in savings each year (based on 150,000 annual labor hours at an average hourly wage of $11.08). The proposal was prepared as a three-point, PERT-type estimate, with the following savings:

	Pessimistic	Most Likely	Optimistic
WIP reduction	35%	50%	60%
Change in Working Capital	$92,400	$132,000	$158,400
Annual Holding Cost Savings	$6,468	$9,240	$11,088
Improved Labor Efficiency	4%	6%	8%
Annual Labor Improvement	$66,480	$99,720	$132,960

The market projections are relatively firm for the next six years, and the rapid appearance of technological innovations in computer products makes it unsound to make projections for periods beyond that time. Therefore, we will use a six year planning horizon for this evaluation. The corporation faces a combined tax rate of 38.7%. The economic factors listed, with a salvage value equals book value

consideration, generate the incremental (challenger process - defender process) cash flows shown in Table 1. The positive terms indicate when the challenger process will have a better return than the defender. The negative values show that there are some conditions where the defender process is better than the challenger process.

Economic factors for the sequential conversion alternative

Because the full conversion alternative has such a wide range of estimated return ($-113,811 to $117,133), the corporation is willing to consider the option where one cell (of the four) is converted immediately, with the final decision for the flexible cell versus machine shop operating process being deferred until the end of one year. A complete listing of all decision options is provided at Table 2. Mixed modes of production will not be acceptable for any time beyond this one year period.

The sequential alternative can be implemented by rearranging on-hand equipment, and deferring the purchase of the remaining equipment until the end of the sample year. However, it is necessary to use $60,000 (of the $76,780) of the conversion expenses for additional wiring, exhaust systems, and other cell setups. The remaining $16,780 will be required for the movement and setup of the last three cells. If the corporation chooses to stay with the machine shop organization at the end of the sample year, there will be a net cost of $6,000 to re-establish that layout.

The second high volume product family was selected to be the sample cell, because its cost and profit results are expected to most closely approximate what the average of the four conversions. If we

Table 1. Annual cash flows and net present value estimates for the time zero (immediate) full cellular conversion alternative.

	PERT-Type Cash Flow Estimate		
Year	Optimistic	Most Likely	Pessimistic
0	$-308,731	$-335,131	$-374,731
1	114,872	93,363	71,287
2	130,814	109,305	87,230
3	113,809	92,300	70,225
4	103,606	82,097	60,022
5	103,606	82,097	60,022
6	95,954	74,445	52,369
NPV(15)*	$117,133	$9,333	$-113,811

*Note: NPV(15) = cash flow net present value, discounted at the corporation MARR of 15%.

Table 2. Decisions, and follow-on options, that were considered at time zero.

Decision	Explanation
d_{00}	Do not initiate any cells, reject this alternative.
d_{01}	Initiate all four cells immediately.
d_{02}	Initiate one cell, collect data, and reevaluate this proposal in one year. (Given that this decision has been selected, the options continue.)
d_{10}	Stop the sequential conversion, revert back to the machine shop layout.
d_{11}	Initiate the remaining three cells.

shift existing assets, then this cell's periodic cash flows and NPV's for each estimate condition are as listed in Table 3. Although the estimated NPV's are each greater than zero (indicating profitability), it must not be forgotten that all equipment must eventually be in a similar configuration. When we include the remaining conversion costs, the new equipment costs, and the one year shifting of the cash flows for the remaining cells, we obtain the estimated NPV's for the sequential approach (these estimates do not use the first cells results for decision input):

Optimistic estimate	$99,135
Most likely estimate	7,945
Pessimistic estimate	-96,928

and the sequential alternative's range of outcomes is less than that of the immediate full conversion alternative.

Development of initial intervals and associated probabilities

To find the expected NPV for each of the alternatives, we must develop their cash flow probability distributions. An important point for these alternatives is that the periodic cash flows for a given estimate are not independent from period to period. For the WIP savings, the reduction level that is attained defines both the one-time return of working capital and the periodic holding cost savings (a percentage of the WIP reduction value). The labor efficiency improvement should remain fairly constant throughout the life of the project. Additionally, as seen in Tables 1 and 3, the ranges of variation between the estimates for years one through six of the full conversion alternative, and years two through six for the sequential alternative are identical. These ranges are offset by depreciation

Table 3. Annual cash flows and net present value estimates for the time zero (immediate) conversion of a single flexible manufacturing cell.

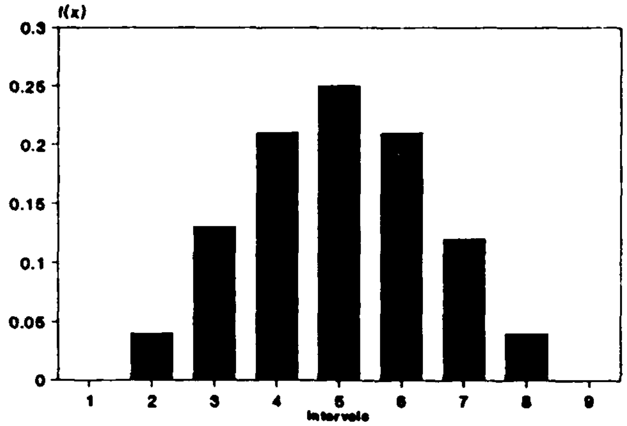
	PERT-Type Cash Flow Estimate		
Year	Optimistic	Most Likely	Pessimistic
0	$2,820	$-3,780	$-13,680
1	22,075	16,698	11,179
2	22,075	16,698	11,179
3	22,075	16,698	11,179
4	22,075	16,698	11,179
5	22,075	16,698	11,179
6	22,075	16,698	11,179
NPV(15)*	$86,364	$59,414	$28,628

*Note: NPV(15) = cash flow net present value, discounted at the corporation MARR of 15%.

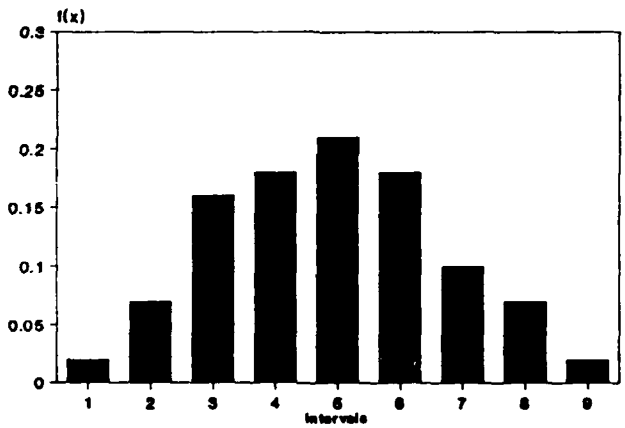
factors, but the relative differences in the estimates are the same. Therefore, when we develop the intervals and their probabilities, we will establish them proportionately for all cash flow three-point estimates.

Since we will be using the first cell's year one results for decision input, we will use those estimates to develop relative intervals, and the probability distribution. We use the three estimates ($22,075, $16,698, and $11,179) to fit a beta distribution [Prueitt and Park (1989(c))]. Our fitting yields a nonstandard beta distribution with shape parameters, α=4.04 and β=3.96. We then establish the discrete intervals, based on the range of outcomes. We will use the range $11,075 to $22,325, in nine intervals of $1,250. We use the incomplete beta function to find the cumulative probabilities within each of the intervals, and graphically represent that value at the midpoints of the respective intervals, as shown in Figure 4(a). After review, the prior beliefs were determined to be more accurately represented by the distribution in Figure 4(b), a flatter, slightly skewed distribution. (These changes preclude further representation as a beta distribution.)

The developed distribution is then applied proportionately to the 3-point estimates of the immediate full conversion, and sequential conversion alternatives. The expected value of the full conversion alternative is $207, found by summing the products of the interval midpoints and their respective probabilities. Under a pure expected value decision criteria, we would accept this alternative because its expected NPV is greater than zero. The expected value of the



(a) Initial distribution, based on 3-point estimates and incomplete beta function.



(b) Refined distribution.

Figure 4. Development of the discrete prior probability distribution.

sequential alternative, <u>ignoring</u> the year one sample result, is \$-131. The drop in the expected value is a reflection of the potential payoff losses due to the postponement of the final decision. The sign change also indicates that the defender layout is superior to the sequential alternative (when the sample information is not used).

<u>Development of initial Dirichlet descriptive parameters</u>

A better approach for the sequential alternative is to use the sample information to update our beliefs about the expected NPV outcome. We first develop the IQF, by making an assessment of the quality of the estimates as compared to one annual cash flow observation. In this study, the prior estimates are felt to be three times as accurate as one annual cash flow observation. We then take this factor of three, multiply it by the respective interval probabilities, θ_j's, to obtain the initial Dirichlet descriptive parameters, α_j's. These θ_j's and α_j's reflect the strength of our prior beliefs.

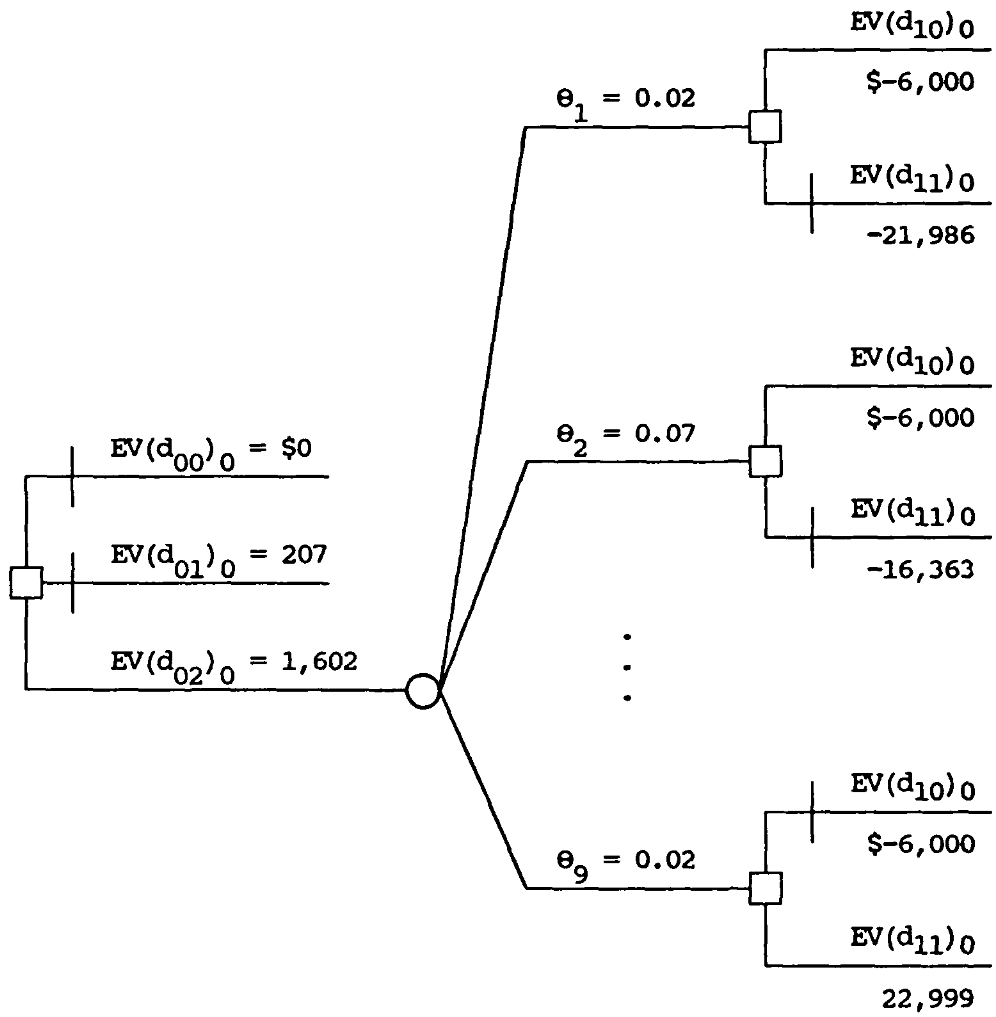
The multinomial predictive distribution, because there is only one sample observation to be recorded, has the same interval probabilities as the prior beliefs, θ_j's. For each predicted outcome, the predicted posterior distribution can be found, and, subsequently, the expected value of that result. For example, for the interval \$11,075 to \$12,325, the predictive probability of an observation falling in that interval is 0.02. Given that this event occurs, the revision process is shown as follows:

Prior α_j	+ Observed x_j	= Posterior α_j	→ Posterior θ_j	→ Interval Expected value
0.05	1	1.05	0.26	$-22,998
0.20	0	0.20	0.05	-3,300
0.48	0	0.48	0.12	-5,128
0.54	0	0.54	0.13	-2,686
0.63	0	0.63	0.16	380
0.54	0	0.54	0.13	3,332
0.31	0	0.31	0.08	3,696
0.20	0	0.20	0.05	3,545
0.05	0	0.05	0.01	1,171
		4.00		$-21,986

This process is then repeated each of the nine intervals. For any sample result, a decision must be made between converting the remaining cells (with their respective expected values) or halting the conversion process (at a net cost of $6,000). This leads to the development of the abbreviated decision tree of Figure 5. (The tree is abbreviated as only the expected value of the conversion payoff is given, instead of illustrating all of its following 9 branches. The term $EV(d_{jk})_t$ is used to represent the expected value of the nine intervals for decision d_{jk}, discounted at the MARR to time t.) In the figure, branches that are cross-hatched (—+—) are inferior decisions. As shown, the sequential approach is superior to the other alternatives, and provides the time zero decision strategy:

Time 0 — Convert the second product family's high volume cell, using existing equipment, and record the first year's results.

Time 1 — If the year one cash flow falls into any of the categories in the range $11,475 to $14,825, then terminate the conversion process. If the cash flow



Note– $EV(d_{jk})_t$ = expected net present value of decision d_{jk}, discounted at the corporation MARR, to time t.

Figure 5. Preposterior analysis of sequential alternative, using one sample observation, and net present values discounted to time zero.

is greater than $14,825, then convert the remaining cells.

The equivalent sample size concept and the sequential alternative

As the sample data were recorded, the economic conditions within and outside the factory were noted. It was observed that there were several differences between the observed conditions and those used in the estimating process. After evaluation, it was felt that the sample data should carry as much informative "weight" as the prior estimates (the Σx_e should equal the IQF of 3). This takes the equivalent sample result outside of those considered in the time zero preposterior analysis. The observed result was $15,500, which falls into the fourth interval. This end of year one prior to posterior probability revision is shown as follows:

Prior α_j	+ Observed x_j	= Posterior α_j	→ Posterior θ_j	→ Interval Expected value
0.05	0	0.05	0.01	$-851
0.20	0	0.20	0.03	-2,530
0.48	0	0.48	0.08	-3,931
0.54	3	3.54	0.59	-13,602
0.63	0	0.63	0.10	291
0.54	0	0.54	0.09	2,554
0.31	0	0.31	0.05	2,834
0.20	0	0.20	0.03	2,717
0.05	0	0.05	0.01	898
		6.00		$-11,620

This revision has an end of year one expected value of $-11,620, which is well below the halt conversion decision payoff of $-6,000. Therefore, our best decision is to halt conversion and revert back to the machine shop layout.

Sensitivity analysis

Because the principle purpose of this case study is to illustrate the techniques we have developed, we will not present a sensitivity analysis of the conventional factors (demand, interest rates, etc.), but focus on an analysis of the factors unique to this technique.

The first factor we examine is the selection of the initial Dirichlet shape parameters, the α_j's. Recall that our initial development considered the estimates' IQF as three, and, thereby, $\Sigma\alpha_j = 3$. We will now vary this IQF value, and observe the effects of the variation on the posterior θ_j's. While it is necessary to examine the results of the observed sample falling into each of the nine intervals, for brevity's sake, we will illustrate this analysis with the observation falling into the fourth interval. Table 4 lists how the posterior θ_j's respond to the observation, for each of the IQF values, with Figure 6 illustrating the trend for the IQF values of one, five, and nine. The figure shows that as the IQF increases, the distribution becomes less responsive to the sample data. When we extend this phenomenon to its effects on the expected NPV, we find that varying the IQF leads to the changes in the payoff, $EV(d_{02})_0$, that are shown in Figure 7. Further, it can be shown that as the IQF approaches infinity, this expected NPV asymptotically approaches \$-131. (A sequential alternative that did not use the sample information would have an identical payoff.) Now, recalling that $EV(d_{01})_0 = \$207$, we see that if the IQF is no more than six, then the sequential option has the greatest payoff. The case study result is sensitive to this factor.

Table 4. Sensitivity analysis of Information Quality Factor (IQF) values ranging from one to nine.

Interval	Prior θ_j	Sample	Posterior θ_j's for IQF values -								
			1	2	3	4	5	6	7	8	9
1	0.02	0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02
2	0.07	0	0.03	0.05	0.05	0.05	0.06	0.06	0.06	0.06	0.06
3	0.16	0	0.08	0.11	0.12	0.13	0.13	0.14	0.14	0.14	0.14
4	0.18	1	0.59	0.45	0.38	0.34	0.32	0.30	0.28	0.27	0.26
5	0.21	0	0.10	0.14	0.16	0.17	0.17	0.18	0.18	0.19	0.19
6	0.18	0	0.09	0.12	0.13	0.14	0.15	0.15	0.16	0.16	0.16
7	0.10	0	0.05	0.07	0.08	0.08	0.09	0.09	0.09	0.09	0.09
8	0.07	0	0.03	0.05	0.05	0.05	0.06	0.06	0.06	0.06	0.06
9	0.02	0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02

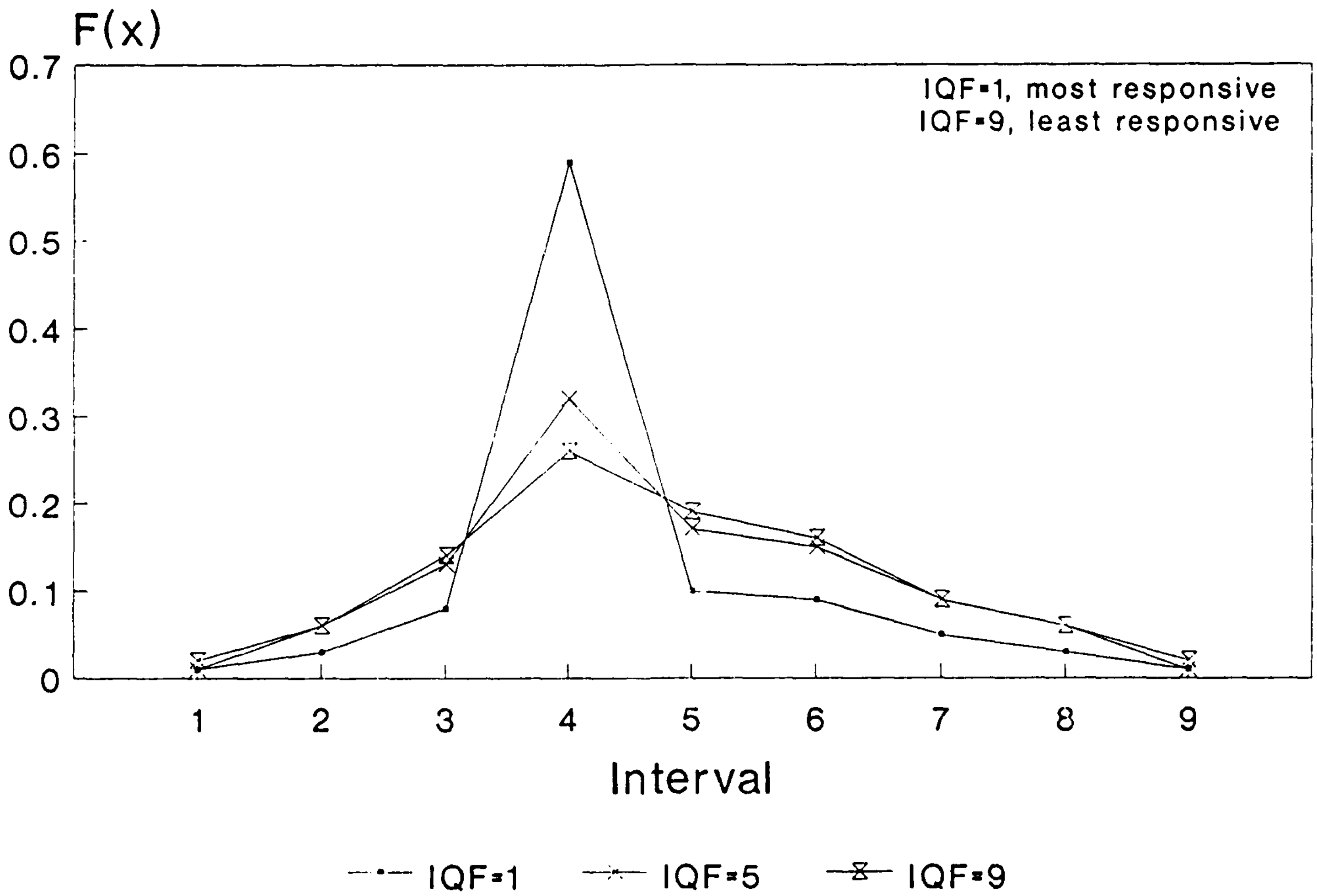


Figure 6. Sensitivity analysis of the Information Quality Factor (IQF).

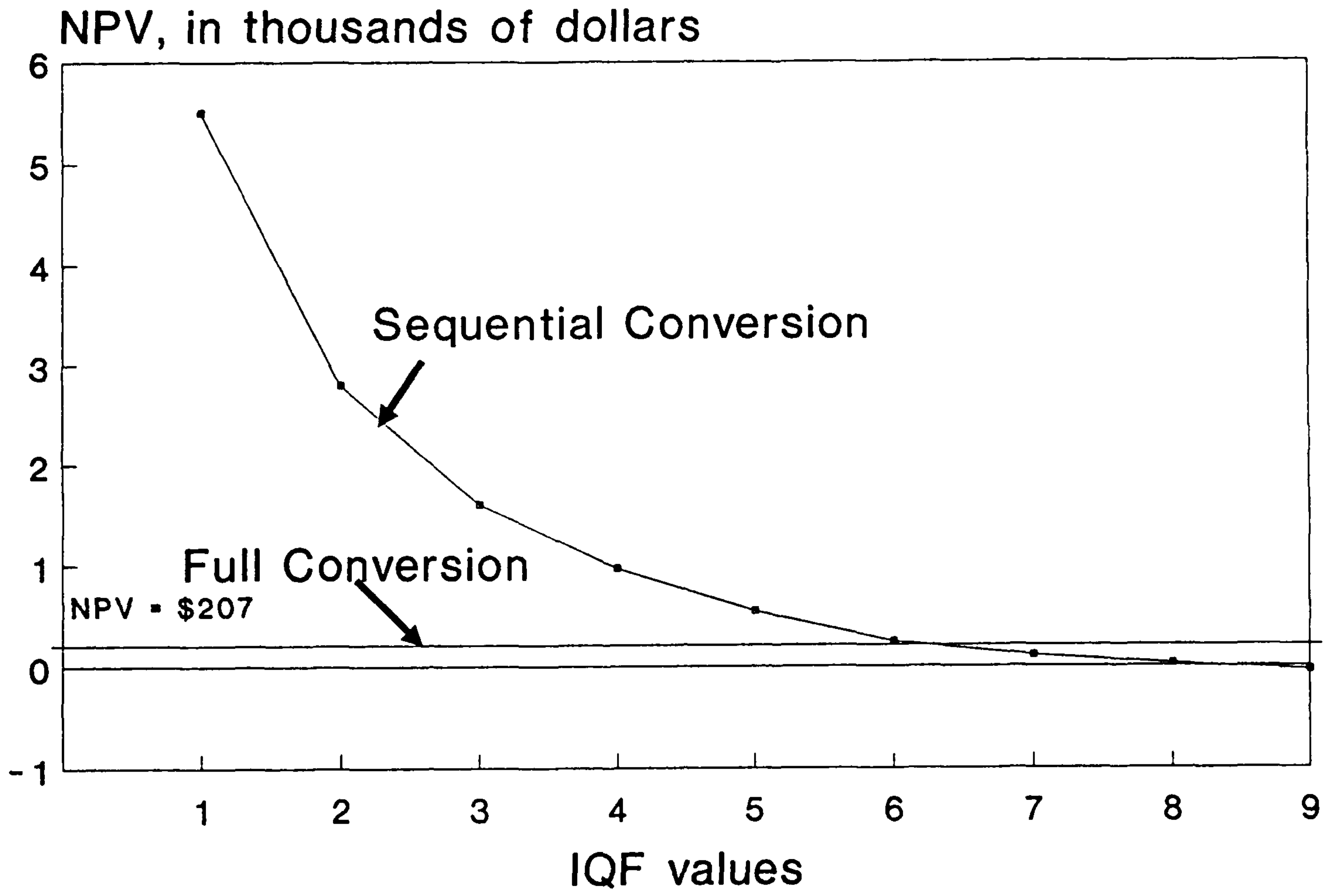


Figure 7. Effects of IQF changes on the net present value of the sequential conversion option.

The sensitivity analysis of the equivalent sample size can be interpreted directly from the IQF analysis. The values of x_e and IQF form a proportional relationship. In the above IQF analysis, the sample size was one observation, and the IQF was some multiple of that value. By using x_e, we change the proportion. In the case study, the use of $x_e=3$ effectively changed the IQF = $\Sigma\alpha_j = 3$, from a three to one, to a three to three relationship. The change in the posterior probabilities is identical to the one that would have occurred if the IQF had originally been one, and one was used as the sample size. It can be shown that similar relationships exist for other proportions.

These analyses show that this model and case study are sensitive to the selections of the IQF and x_e values. From a philosophical perspective, it shows that the stronger the belief, the less likely sample results will change any initial decisions. In fact, if we feel we have no uncertainty (implicitly allowing IQF to go infinity), then we have the known distribution condition that is unaffected by any sample results. However, when uncertainty exists, particularly when there is a large amount of uncertainty, this technique provides a responsive model that takes advantage of any sample information.

CONCLUSIONS

In this paper, we have developed a technique that can be used to provide uncertainty resolution to the expected net present value of a project's cash flows. We have shown how any form of prior beliefs, when approximated by a discrete distribution, can be used to develop a Dirichlet distribution, and how that distribution's descriptive

parameters can be used to reflect our perceived quality of those initial beliefs (through the use of the IQF). We then show how unforeseen conditions in the sample observations can be taken into consideration during the revision process, through the use of the equivalent sample size concept.

We then exercised these concepts in a case study analysis of an actual decision problem. The most important point of that example was that in the time zero decision strategy, we directed that if the sample observation was greater than $14,825, then we should convert the remaining cells. However, using the equivalent sample size approach changed that strategy. The observed result, $15,500 with a $x_e = 3$, generated a revised distribution whose expected net present value was different from those predicted at time 0. This led us to a modified decision strategy that directed halting the conversion process. Therefore, the ability to generate a revised decision strategy, based on the most accurate information, will permitted us to increase the firm's profitability.

The sensitivity analysis shows that the IQF and x_e techniques provide responsiveness to the model, which is particularly beneficial in decision situations that have uncertainty. It also shows that these values must be selected carefully, to preclude any inappropriate or biased results.

These methods should be applicable to many other situations, with similar decision improvement results.

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VI. MONITORING PROJECT PERFORMANCE WITH POST-AUDIT INFORMATION: CASH FLOW CONTROL CHARTS

ABSTRACT

An important aspect of the capital budgeting process is the post-audit, which involves (1) comparing actual results with those predicted by the decision maker and (2) explaining why any differences occurred. When decision makers systematically revise their uncertain initial forecasts with actual outcomes, there is a tendency for the estimates to improve. As any biases are observed and eliminated, management can improve operations and bring results and forecasts into agreement. This paper presents a post-audit method for the class of investment problems where each element of the cash flow forecast is uncertain. These problems have multiple, identical units with uncertain cash flow estimates, as found in many fleet replacement problems or in advanced manufacturing systems with multiple cells. This method graphically illustrates the uncertainty resolution that occurs, providing a means to post-audit and monitor the performance of a project through the development of Cash Flow Control Charts.

INTRODUCTION

The capital budgeting process embraces a rather broad and diverse class of activities in allocating capital resources to competing investment projects. While phases in the decision making process could be identified in many ways, we divide it into four: (1) identification of areas of opportunity, or problems, that indicate a need for capital expenditure, (2) development of various projects in response to the opportunity or need, (3) selection of projects for implementation, and (4) control or evaluation of the performance of the approved projects.

The development phase, which includes definition and cash flow generation, is often considered as the most difficult portion of the capital budgeting process. The success of this phase depends on the type and availability of information provided for the capital budgeting process. The selection phase encompasses such things as measures of investment worth, the timing of the investment, the determination of the amount to be invested within any given time period, and arranging for the financial means necessary for the completion of the projects. In contrast, the major concern in the control phase relates to the improvement of future capital expenditure, by learning from past results. This is accomplished through the use of post-audits. It is a common practice to post-audit projects, and provide feedback to decision makers and analysts about the accuracy of the forecasts used to make these decisions. In this paper, we are primarily concerned with the analytical techniques used during the post-audit phase.

The need for a post-audit can be critical for investments in advanced manufacturing systems. Many companies must maintain capacity

requirements and keep pace with newer competitors by upgrading their manufacturing processes with computers, robotics, and other sophisticated production processes. These advanced manufacturing systems frequently have high investment costs, but do <u>not</u> have extensive performance "track records" (as compared with more traditional processes). Further emphasizing the need for accurate forecasts is that, as they strive for optimal capital utilization, many firms are including implemented projects as nonliquid assets in their capital budgets, in the amounts of their salvage, or abandonment values [13]. The post-audit will affect both the implemented project and subsequent budgeting decisions.

We must recognize that each element of the cash flow forecast is subject to uncertainty, so a percentage of all projects undertaken by any reasonably venturesome firm will necessarily go awry. Whenever the actual outcomes for the project implemented differ from the forecasts, management begins to worry about making decisions good (as opposed to making good decisions). Management's problems are further compounded by prolonged beliefs that the initial estimates are true distribution values. The differences are usually not readily detected because little emphasis is given to post-audit information.

The concept of utilizing conventional quality control charts with economic factors was first used as an attempt to monitor and minimize resource usage. These initial works did not consider cash inflows, but focused on physical inputs, such as labor hours, production units, and raw materials used [9,17]. These initial efforts treated the design parameters as true distribution values, and used these charts to

control the expenditure process. Subsequently, the sources of possible cost variation were addressed: (a) controllable versus noncontrollable variation, (b) investigative cost viability, and (c) overlapping, or joint variation causes [12]. These discussions focused on keeping expenditures within the designed budget. Other methods sought to provide new graphical measures of effectiveness [14], but these methods fail to consider cash inflows and the time-value-of-money concept.

A different approach uses Bayes theorem [6,22], and recent surveys indicate that classical and empirical Bayesian methods (where prior beliefs are based only on past empirical results) are being more widely used in decision making areas [15,26]. Bayesian techniques have been applied to modular projects, under sequential sampling, with periodic project continue/discontinue decision making [4,5]. A previous work developed a one step, project balance control chart using Bayesian methods [18]. Other works applied Bayesian methodology as a means to correct the problem of given an abnormality, should we investigate or not investigate [9], and the revision of expected machine downtime and number of control inspections [12].

The purpose of this paper is to develop a systematic means to post-audit, monitor, and revise the initial estimates of an on-going project, <u>and</u> graphically portray this information to the user. This will be accomplished by developing Cash Flow Control Charts (CFCC), which are a unique combination of Bayesian revision techniques and statistical quality control methods. At this point, we want to emphasize that these CFCC are <u>not</u> Shewhart-type control charts ($\bar{x}$ and R-charts). They are similar in appearance, because the CFCC use

some of the same procedures for initial set-ups, but the CFCC differ markedly in their purpose and execution procedures.

DEVELOPMENT OF CASH FLOW CONTROL CHARTS

To explain our development of cash flow control charts, we will first discuss the underlying assumptions, other factors that need to be considered, the conceptualization of the CFCC, and their limitations.

Assumptions

The methods we are about to describe and use in the illustrative example are applicable under the following assumptions:

1. The CFCC are designed with the assumption that the project being post-audited has some measure of repeatability, a situation that frequently occurs with "fleet" replacement problems. In this situation, the incoming equipment fleet has multiple like items (whether they are many load carrying vehicles or a just a few manufacturing machines). This permits the project to be divided into several identical cells.

2. The cash flows from manufacturing equipment performance are assumed to be relatively stable (as compared to the stock market, for example), because the equipment is being used in an environment that has existing, as well as projected, production requirements. For the specific example we have included in this paper, the cash flow forecasts will not have trend or seasonal effects. (Though not covered in this text, we believe that this assumption can be relaxed, to address problems with those elements. These problems would require that the cash flow residuals, observations minus forecasts, be IIDRV.

This extension will be more apparent, after examination of the concepts presented here.)

3. The cash flows from these cells are assumed to be independent identically distributed random variables (IIDRV). Thereby permitting the Central Limit Theorem to be applied to the cash flow performances of groups of these cells [25].

4. We assume that we can apply appropriate conventional quality control chart procedures for subgroups, subgroup size, and sampling methods [2,7].

Other Factors to Consider

Before developing our CFCC, we need to address the following additional factors: (1) time value of money, (2) timing of sampling and (3) cost of sampling.

Time-value-of-money. For cash flow realizations, an important factor that may influence the sampling method is the time-value of money. Machining processes can be generalized as generating products, on a regularly timed basis, for operating periods of hours, shifts, or days. These products, once produced and inspected, experience very small changes in value over time (usually only in the form of inventory holding costs). Project cash flows are usually projected over longer time periods (weekly, monthly or annually), and they usually have some variability associated with their projected time of occurrence. Thus, some otherwise identical cells, but with different expense and revenue schedules, will have different net values when they are discounted to a common decision point in time.

Time periods of observation. If the individual cells come into service at different points in time, then decisions must be made to determine if it is more important to capture each respective cell's cash flow information over a common period in its economic life (say first year of operation), or if it is more important to capture cash flow information over a common operational period (say, January to May). For common economic life periods, endogenous factors, such as tool wear or operator experience, are homogeneous. Common operational periods will have homogenous economic environmental conditions. We can also require that the internal performance and external conditions be homogenous from period-to-period. This can be done by making each subgroup consist of all cell performances in a specific period (say, a week), and then use a different period (say, the next week) for the next subgroup. Thereby, the control chart horizontal axis will delineate different time periods, instead of physically different subgroups.

When period-to-period sampling is used, the method requires the somewhat limiting assumption that the economic environmental conditions are identical from period-to-period. However, this sampling method can be combined with period-by-period Bayesian revision, and provide uncertainty resolution faster than conventional control chart procedures.

Cost of Sampling. Conventional control charts consider sample size and cost of inspection. However, these cost limitations do not normally apply to cash flow control charts, because the cash flow data

is already being collected for tax and stockholder reports. So, it is essentially free sample information.

Conceptualization of the Cash Flow Control Charts

Before going further, we need to stress the differences in rationale behind conventional quality control charts and these CFCC. When a conventional control chart indicates that the process is "in control", it means that the process is stable, or predictable, at some predetermined probability level. Upper control limit (UCL) or lower control limit (LCL) violations signal that it is necessary to investigate for causes that may be responsible for the out-of-control condition. The probable cause may be considered random, and no adjustments are made in the process, or there may be assignable causes, and corrective actions are required. When the CFCC is "in control", it means that the cash flow observations are accurately fitting the probabilistic descriptions given by the current estimates. Conversely, UCL or LCL violations indicate that the current estirates ($\mu_{\bar{x}}$ and $\sigma_{\bar{x}}^2$) are inaccurate, and revisions are needed.

This cash flow control chart approach is conceptualized with two charts, one for expected performance and one for performance variability (similar to $\bar{x}$ and R charts, respectively).

CFCC for Expected Performance. As a review, the centerline for the conventional $\bar{x}$-chart is computed from:

$$\bar{\bar{x}} = \sum_{j=1}^{g} \bar{x}_j/g$$

using sample data, where,

$\bar{\bar{x}}$ = average of the subgroup averages

$\bar{x}_j$ = average of the jth subgroup

g = number of subgroups

In operation, the chart's centerline and limits are revised when there is evidence that assignable causes exist, and they have caused the process average to shift.

We alter these procedures for our CFCC for expected performance. Sample data may not be available for the first observation, so the centerline is set at the expected value of the subgroups' mean ($E(\mu_{\bar{x}})$). The UCL and LCL are set as multiples of the expected value of the subgroup variance ($E(\sigma_{\bar{x}}^2)$). The limits may be $\pm 2\sigma_{\bar{x}}$, $\pm 3\sigma_{\bar{x}}$, or some other desired accuracy level. For this paper, we will use $\pm 3\sigma_{\bar{x}}$.

How well the data observations fit the expected value control chart upper and lower limits describes the accuracy of distribution estimates. However, there is a need for an additional term that is only partially analogous to control or reject limits, and that term will be called a minimum return limit (MRL). In basic terms, a simple investment project consists of an initial outlay of capital, followed by a series of cash inflows. When these receipts are discounted at some minimum acceptable rate of return (MARR), and combined with the initial investment, the result must have a net worth of at least zero for the project to be acceptable. The amounts of the receipts thus have an associated minimum acceptable value.

For example, a company has a 2-year project that costs $1000 per producing unit, and their MARR is 10%. The annual return, for particular unit, must be $577 for that unit's performance to be acceptable. The company wants this "break-even" performance level to

appear on the CFCC. Now, because the CFCC works with grouped, rather than individual data, this per unit minimum return must be similarly converted, before it is posted as the subgroup's MRL. This conversion is accomplished by relating the probability that an individual u. t from the parent population will fail to meet the required return, to a corresponding probability that a subgroup average will fail to meet a value (to be determined) for the distribution of subgroup averages. That value is the MRL. This concept can be applied when the producing units' cash flows are IIDRV, regardless of the specific nature of the parent population, because the MRL value is dependent upon the distribution of the <u>subgroup</u> averages (which is normal).

As a simplified demonstration of this concept, we will assume that the estimated annual cash flow for an individual unit has a uniform distribution between \$550 and \$600 (estimates may also be described by beta, normal, or any other distribution). Then, the probability that a particular unit will fail to meet the required level of return, \$577, is

$$P(x < \$577) = (577 - 550) / (600 - 550) = 0.54$$

The parent population variance is

$$\mathrm{Var}(x) = (600 - 550)^2 / 12 = 14.434^2$$

which, for subgroup size of four, gives the subgroup variance

$$\mathrm{Var}(\bar{x}) = \mathrm{Var}((\sum_{j=1}^{g} x_j) / g)$$

$$= 14.434^2 / 4 = 7.217^2$$

Then, the value of the MRL can be found by

$$P(x < \$577) = P(\bar{x} < \mathrm{MRL})$$

$$0.54 = P(Z < 0.1005)$$

so,

$$(MRL - 575) / 7.217 = 0.1005$$

$$MRL = \$575.73$$

which is posted on the chart, and observations falling below this limit are of special concern to the project managers.

<u>CFCC for Performance Variability</u>. The initial set-up of the performance variability control chart is similar to the conventional R-chart, the centerline is computed from:

$$\bar{R} = \sum_{j=1}^{g} R_j/g$$

where,

g = number of subgroups

$\bar{R}$ = average of the subgroup ranges

R_j = range of the jth subgroup

However, where the conventional R-chart uses sample data or a design "standard" to provide the estimate of the parent population's standard deviation, we will use the expected value of the standard deviation of the estimated cash flow. Thereby, using $\pm 3\sigma_x$ accuracy limits:

$$UCL_R = d_2\sigma_x + 3d_3\sigma_x = D_2\sigma_x$$

$$LCL_R = d_2\sigma_x - 3d_3\sigma_x = D_1\sigma_x$$

where,

d_2, d_3, D_1 and D_2 are standard table values [7], mathematically based on the subgroup sample size, n, and

σ_x is the expected value of the population standard deviation, $(E(\sigma_x^2))^{\frac{1}{2}}$.

For a conventional R-chart, the user is cautioned about using design values to determine the UCL and LCL. As specification errors that make σ_X too small lead to unwarranted expenses, as attempts are made to determine assignable causes of error when only chance causes may be at work, and, if the σ_X specification is too large, the process will appear to be in control when there are actually deviations with assignable causes present. Our CFCC for performance variability does not have these specification concerns, because its purpose is <u>not</u> to monitor if the observations are within the charts parameters, but is, instead, to use the observations to revise its chart parameters.

Updating the expected values for the subgroup mean and variance will require the CFCC centerlines and control limits be redrawn after each revision (creating more frequent reconstructions than conventional procedures). As the observations move from an out-of-control to an in-control condition, we can conclude that uncertainty resolution is occurring and our post-audit estimates are more accurate than our initial estimates.

<u>Limitations</u>

When working with the expected performance control chart, (as with an $\bar{x}$-chart) the applicability of the techniques is based on the observations being IIDRV's, such that the subgroup means are normally distributed, as shown in Figure 1. The concern is that in focusing on the grouped data, there is no inherent procedure that will work backwards to describe the underlying parent population. To be specific, just because an observed $\mathrm{Var}(\bar{x})$ matches its estimate, it does not necessarily mean that the assumed distribution is correct. For

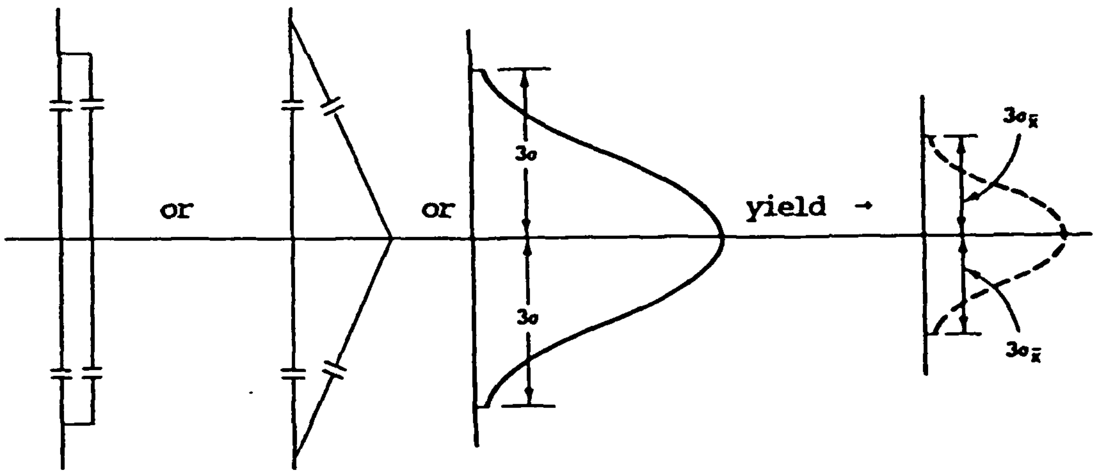


Figure 1. Examples of distributional relationships for parent populations of individual values and subgroups' mean values.

example, an observed sample, in groupings with n = 4, shows $\bar{x} = 5$ and $\mathrm{Var}(\bar{x}) = \sigma_{\bar{x}}^2 = 5^2$. The prior belief was that the parent population was normal, and, if true, its variance given by:

$$\mathrm{Var}(x) = \sigma_x^2 = n\sigma_{\bar{x}}^2 = 4 * 5^2 = 100 = 10^2$$

Following this belief, the distribution is believed to be $N(5,10^2)$. However, it is also possible that the same $\mathrm{Var}(\bar{x}) = 5^2$ could have come from some other type of distribution. It could have come from a uniform distribution, so that

$$\mathrm{Var}(\bar{x}) = \mathrm{Var}(x_{\mathrm{uniform}}) \,/\, n$$

and, recalling

$$\mathrm{Var}(x_{\mathrm{uniform}}) = (\mathrm{high} - \mathrm{low})^2 \,/\, 12$$

then,

$$n * \mathrm{Var}(\bar{x}) * 12 = (\mathrm{high}-\mathrm{low})^2$$

$$(4 * 5^2 * 12)^{\frac{1}{2}} = 20\sqrt{3} = \mathrm{high} - \mathrm{low} = \text{range of } x$$

and, x is distributed as

$$\approx U((\mu - 10\sqrt{3}),\ (\mu + 10\sqrt{3})) = U(-12.3205,\ 22.3205)$$

The importance of this realization goes back to the initial project selection. There, consideration was given to the project's potential for success or failure, based on its probability of net negative return, Prob $(x < 0)$. Using the normal prior belief, it could be concluded that the $N(5,10^2)$ distribution has

$$P(x < 0) = P\left(\frac{(x-5)}{10} < \frac{(0-5)}{10}\right) = P(Z < -0.5) = 0.3085$$

but, using the uniform distribution possibility

$$P(x < 0) = \frac{(0 - (-12.3205))}{(22.3205 - (-12.3205))} = 0.3557$$

a somewhat different result, and one that is obscured in standard $\bar{x}$-chart procedures.

If the mean and variance (the first and second moments) of the observed and forecast distributions are the same, an examination of the third central moments of the distributions can confirm if the two distributions are, indeed, identical. If these moments differ, we will know that our observed distribution is not what we predicted, and we may not be able to clearly determine the observations parent distribution.

Further, when working with the performance variability chart (as with the R-chart), we know that the LCL_R, $centerline_R$, and UCL_R formulas are based on the assumption that the parent population is normally distributed, but minor departures from normality have not created major difficulties [7]. However, the central purpose of this technique is to refine the accuracy of the cash flow distribution, so we will attempt to overcome this limitation by conducting goodness-of-fit testing (in addition to the independence testing) as we construct the cash flow control charts.

The development of the MRL implies the use of a uniform series of cash flows, as each subgroup average is compared to this fixed limit. This does not infer that the cash flow performances must be uniformly distributed, nor does it infer that the individual cash flows must be identical. Instead, this implies that we anticipate that the subgroup average cash flows are more likely to have nearly the same value, from period-to-period, than other investments. This is a reasonable assumption, because the cash flows from equipment replacement projects

tend to be more stable than other types of investments. In fact, survey results show that 57-percent of the firms polled had cash flow estimation errors at less than 10-percent for equipment replacement projects, as compared to 43-percent for all other investments [21]. If the cash flow estimates are an uneven series of payments, these CFCC can still be used for updating purposes. However, their construction may require the use of the residuals (subgroup average less estimate) of the cash flows, rather than the observations themselves. Another alternative would be to transform the data to some approximately normal distribution [23]. Thereby, when considering these factors, as well as the time-value-of-money, if we do proceed with the MRL, we would exercise caution in its use and interpretations.

UPDATING CASH FLOW CONTROL CHARTS - BAYESIAN REVISION

When the cash flow realizations are correctly described as forming a random sample from a normal distribution, this distribution has two parameters of interest, the subgroup mean ($\mu_{\bar{X}}$) and variance ($\sigma_{\bar{X}}^2$). Two important scenarios are:

1. The mean cash flow for the subgroup is unknown, but the variance is known.
2. Both the cash flow mean and variance are unknown.

The appropriate analytic approach will be determined by the level of estimate detail included in the proposal. When $\mu_{\bar{X}}$ is unknown, it will have an associated expected value, $E(\mu_{\bar{X}})$, and some variance, $Var(\mu_{\bar{X}})$. The calculation of $E(\mu_{\bar{X}})$ will involve all factor inputs at their most likely outcomes. The $Var(\mu_{\bar{X}})$ is determined from the variability of the

inputs' most likely outcomes. This must not be confused with the $\sigma_{\bar{x}}^2$ terms, which are based on the <u>entire</u> range of outcomes for the inputs (pessimistic to optimistic). The value of $E(\sigma_{\bar{x}}^2)$ can be thought of as being developed from the most likely ranges of input outcomes, while $Var(\sigma_{\bar{x}}^2)$ describes the variation in those ranges. Now, proposal estimates may be written in terms of the individual producing units, rather than as subgrouped estimates that have the necessary normal distribution properties. If so, it will be necessary to use the following transformations (which assume that the covariance terms are equal to zero):

$$E(\mu_x) = E(\mu_{\bar{x}}) \tag{1}$$

$$E(\sigma_x^2) = E(n * \sigma_{\bar{x}}^2) = n * E(\sigma_{\bar{x}}^2) \tag{2}$$

$$Var(\sigma_x^2) = Var(n * \sigma_{\bar{x}}^2) = n^2 * Var(\sigma_{\bar{x}}^2) \tag{3}$$

$$Var(\mu_x) = \sigma_{\mu_x}^{\ 2} = n * \sigma_{\mu_{\bar{x}}}^{\ 2} \tag{4}$$

Where n is the subgroup sample size. The importance of this reasoning stems from the survey results that show increasing numbers of firms using probability theory in their cash flow estimations, with 44% of the larger, more capital intensive firms already using some form of range estimations [21].

<u>Population Mean Unknown, Variance Known</u>

Here, the population variance of the cash flow is known, or assumed known, due to the following circumstances:

1. The value of σ_x^2 is known and given in the project proposal.
2. The project proposal fails to provide enough information to determine $Var(\sigma_x^2)$, so the estimate of $E(\sigma_x^2)$ is assumed as

known ($E(\sigma_X{}^2)$ assumed to equal $\sigma_X{}^2$), and the term $\mathrm{Var}(\sigma_X{}^2)$ is not used. (This includes those situations where $\sigma_{\bar{X}}^2$ is approximated by some $s_{\bar{X}}^2$.)

The subgroups' mean is unknown, and the uncertainty about this parameter is assumed to follow a normal probability distribution, with some estimated mean and variance. The subgroup means provide a normal sampling process, and the necessary structure for the normal distribution's family of natural conjugate distributions.

Specifically, the prior normal distribution of $\mu_{\bar{X}}$, combined with a normal sampling procedure, produces a posterior distribution of the belief in $\mu_{\bar{X}}$ that is also normally distributed. The Bayesian revision begins with the initial estimates of $E(\mu_{\bar{X}})$ and $\mathrm{Var}(\mu_{\bar{X}})$, represented by the notation for the <u>prior</u> <u>beliefs</u>:

$$E(\mu_{\bar{X}}) = m' \qquad \text{and} \qquad \mathrm{Var}(\mu_{\bar{X}}) = \sigma'^2$$

(the ' denotes prior beliefs). The sampling information is taken from g subgroups, with overall average, m, and the posterior parameters (the " denotes posterior beliefs) are found by [22]:

$$\frac{1}{\sigma''^2} = \frac{1}{\sigma'^2} + \frac{g}{\sigma_{\bar{X}}^2} \tag{5}$$

and,

$$m'' = \frac{\dfrac{m'}{\sigma'^2} + \dfrac{gm}{\sigma_{\bar{X}}^2}}{\dfrac{1}{\sigma'^2} + \dfrac{g}{\sigma_{\bar{X}}^2}} = \frac{m'\sigma_{\bar{X}}^2 + gm\sigma'^2}{\sigma_{\bar{X}}^2 + g\sigma'^2} \tag{6}$$

with these <u>posterior</u> <u>values</u>, we have the updated estimates

$$E(\mu_{\bar{X}}) = m''$$

$$\mathrm{Var}(\mu_{\bar{X}}) = \sigma''^2$$

Since $\sigma_{\bar{x}}^2$ is fixed, we see that in equation (5), that as g approaches infinity, the Var($\mu_{\bar{x}}$) goes to zero. We interpret this as complete uncertainty resolution of the mean, since the population of samples is consumed. A short example will demonstrate this process.

A hypothetical firm has just replaced a dozen drill presses with new ones that are estimated to save \$100 per month, but that estimate is uncertain, and has a variance, Var($\mu_{\bar{x}}$) = 15^2. The savings variation from unit-to-unit is known, $\sigma_x^2 = 10^2$. One month's cash flow data is used to monitor project performance and update the monthly savings estimate. The twelve presses are randomly assigned to subgroups of four units each. At the end of the month, the three subgroups combined average savings was \$85 per unit. The estimates were revised by using equations (5) and(6):

$$\sigma_{\bar{x}}^2 = \frac{10^2}{4} = 25 = 5^2$$

$$\frac{1}{\sigma''^2} = \frac{1}{15^2} + \frac{3}{5^2} = 0.124$$

so,

$$\mathrm{Var}(\mu_{\bar{x}}) = \sigma''^2 = 8.036 = 2.835^2$$

$$E(\mu_{\bar{x}}) = m'' = \frac{100(5^2) + 3(85)(15^2)}{5^2 + 3(15^2)} = 85.536$$

The posterior belief is that each drill press' monthly savings are normally distributed as N(\$85.54, 10^2), and the Var($\mu_{\bar{x}}$) is reduced to 2.835^2.

Population Mean and Variance Unknown

A more likely scenario considers the mean ($\mu_{\bar{x}}$) and variance ($\sigma_{\bar{x}}^2$) of this distribution as unknown. This situation requires the assignment of a joint prior distribution of $\mu_{\bar{x}}$ and $\sigma_{\bar{x}}^2$, with subsequent determination of an appropriate posterior distribution, given an observed sample. The natural conjugate distribution family that satisfies these circumstances is the normal-gamma family of distributions. This family assumes that the conditional distribution of $\mu_{\bar{x}}$, for a given value of $\sigma_{\bar{x}}^2$, is normal, with a variance that is proportional to $\sigma_{\bar{x}}^2$. Some authors develop this family by using the term $\tau = 1/\sigma^2$ for computational convenience [6], and the proportional variance is represented as

$$k\tau = k/\sigma^2$$

where k is the scaling constant. Further, the marginal distribution of the process variance reciprocal, $\tau = 1/\sigma_{\bar{x}}^2$, is a gamma distribution. When the number, g, of subgroup means, $(\bar{x}_1,\ldots,\bar{x}_g)$, form a random sample with mean $\mu_{\bar{x}}$, and there is a specific value for τ, the development of a joint, normal p.d.f. is possible as

$$f(\bar{x} \mid \mu_{\bar{x}}, \tau) = (\tau/2\pi)^{g/2} \exp[-(k\tau/2)\Sigma(\bar{x}_j - \mu_{\bar{x}})^2]$$

and the gamma distribution of τ has the p.d.f.

$$f(\tau \mid \alpha, \beta) = \begin{cases} (\beta^\alpha/\Gamma(\alpha))\ \tau^{\alpha-1} \exp[-\beta\tau] & \text{for } \tau>0 \\ 0 & \text{for } \tau\le 0 \end{cases}$$

where,

α and β are the gamma distribution shape parameters, and

$E(\tau) = E(1/\sigma_{\bar{x}}^2) = \alpha/\beta$ (sometimes written as $= 1/v$)

$Var(\tau) = Var(1/\sigma_{\bar{x}}^2) = \alpha/\beta^2$ (sometimes written as $= 2/\delta v^2$)

In the normal-gamma family of conjugate distributions, the terms $\mu_{\bar{X}}$ and $\sigma_{\bar{X}}^{2}$ are dependent, and it is not possible to find the joint prior distribution by simply determining the individual marginal distributions of $\mu_{\bar{X}}$ and $\sigma_{\bar{X}}^{2}$, respectively, and multiplying them together (as appropriate for a single parameter). However, it has been shown that the marginal distribution of $\mu_{\bar{X}}$ is a t-distribution with 2α degrees of freedom [22]. More importantly the two marginal distributions share shape parameters such that

$$E(\mu_{\bar{X}}) = m \qquad \text{if } \delta > 1$$

$$\mathrm{Var}(\mu_{\bar{X}}) = \delta v \;/\; (k(\delta - 2)) \qquad \text{if } \delta > 2$$

where,

δ and v are the shape parameters shared by both the t and gamma distributions, and

k is the scaling constant described in the conditional distribution.

Next, we will examine the relationship between τ and $\sigma_{\bar{X}}^{2}$. Proposal variance estimates will be made in terms of $E(\sigma_{\bar{X}}^{2})$ and $\mathrm{Var}(\sigma_{\bar{X}}^{2})$, and not $E(\tau)$ and $\mathrm{Var}(\tau)$. Therefore, it is critical that the gamma distribution shape parameters be determined by their direct relationship to the proposal estimates. The gamma distribution is described as

$$\int_0^\infty f(\tau \mid \alpha, \beta)d\tau = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} e^{-\beta\tau} \, d\tau$$

with

$$E(\tau) = \frac{\alpha}{\beta} \qquad \text{and,} \qquad \mathrm{Var}(\tau) = \frac{\alpha}{\beta^2}$$

Then, making the variable transformations

$$x = \sigma_x^2$$

$$\tau = 1/x$$

$$\left| \frac{d\tau}{dx} \right| = \frac{1}{x^2} \qquad \text{or,} \qquad d\tau = \frac{dx}{x^2}$$

$$\alpha = \delta/2 \qquad \text{and,} \qquad \beta = \delta v/2$$

so,

$$\int_0^\infty f(\tau \mid \alpha, \beta) d\tau = \int_0^\infty f(1/x \mid \delta, v) dx$$

$$= \int_0^\infty \frac{(\tfrac{1}{2}\delta v)^{\frac{1}{2}\delta}}{\Gamma(\tfrac{1}{2}\delta)} \left(\frac{1}{x}\right)^{\frac{1}{2}\delta - 1} e^{-\delta v/2x} \frac{dx}{x^2}$$

$$= \int_0^\infty \frac{(\tfrac{1}{2}\delta v)^{\frac{1}{2}\delta}}{\Gamma(\tfrac{1}{2}\delta)} \left(\frac{1}{x}\right)^{\frac{1}{2}\delta + 1} 2e^{-\delta v/2x} dx$$

which is an inverted-gamma distribution [22]. This distribution has

$$E(x) = E(\sigma_x^2) = \frac{\delta v}{\delta - 2}$$

$$\text{Var}(x) = \text{Var}(\sigma_x^2) = \frac{2\delta^2 v^2}{(\delta-2)^2(\delta-4)}$$

So, with this information, it is possible to determine the prior distribution shape parameters directly from proposal estimates, and to describe the prior to posterior parameter relationships. The values for m', k', v', and δ' are found from the relations

$$E(\mu_x) = m' \tag{7}$$

$$\text{Var}(\mu_x) = \frac{\delta' v'}{k'(\delta'-2)} \tag{8}$$

$$E(\sigma_x^2) = \frac{\delta' v'}{(\delta'-2)} \tag{9}$$

$$\text{Var}(\sigma_x^2) = \frac{2\delta'^2 v'^2}{(\delta'-2)^2(\delta'-4)} \tag{10}$$

Then the posterior distribution can be found, with the incorporation of sample information, by

$$m'' = \frac{(k'm' + gm)}{(k' + g)} \tag{11}$$

$$k'' = k' + g \tag{12}$$

$$\delta'' = \delta' + g \tag{13}$$

$$v'' = \frac{\delta'v' + k'm'^2 + (g-1)v + gm^2 - k''m''^2}{\delta' + g} \tag{14}$$

where,

g is the number of subgroup means,

m is the mean of all subgroups, and

v is the sample variance, from $(1/(g-1))\ \Sigma(\bar{x}_j - m)^2$.

Then, when the posterior descriptive parameters have been calculated, they become the prior descriptive terms for the next iteration.

ILLUSTRATIVE EXAMPLE

As a demonstration of the concepts developed in this paper, the subsequent section provides a concise summary of the procedures, followed immediately by an example problem.

Cash Flow Control Chart Methodology

The procedures explained in the development and updating sections are summarized as follows:

1. Determine the expected value and variance of the underlying distribution of individual observations by using the project proposal estimates and the appropriate formulas for that type of distribution.

2. Determine the desired subgroup size, n, for the conditions, or situation of interest.
3. Determine the predicted standard deviation of the subgroup means.
4. Construct the expected performance control chart. Determine the desired description accuracy for the subgrouped means ($\pm 3\sigma_{\bar{X}}$, $\pm 2\sigma_{\bar{X}}$, or any other desired accuracy level), and post the UCL and LCL, respectively. Determine any minimum return limits (MRL), and post them accordingly.
5. Construct the performance variability control chart, using the predicted value of σ_X and the desired accuracy level to determine the control limits.
6. Determine the prior distribution shape parameters m', d', v', and δ'.
7. Post the observations. Determine if there are any control limit violations, and, if so, determine if these violations have assignable causes.
8. Revise the parameter estimates for $\mu_{\bar{X}}$ and $\sigma_{\bar{X}}^2$, using Bayesian revision techniques. Use the revised expected values to compute new control limits for each chart, in preparation for the next set of observations. Provide the updated estimates to the appropriate sequential decision making review, in case there is a need to change current decision strategies.
9. Test observations for independence and normality characteristics.

10. Repeat steps until project terminates.

These procedures can be easily automated on a computer.

Example with Sample Data

A firm has selected four type A forging presses to replace its current fleet. The project proposal claims that the monthly operating cost savings will be \$50,000 per unit. It is recognized that there is a degree of uncertainty associated with the estimated savings, uncertainty that is described by the parameters (numbers scaled as thousands):

$$E(\mu_X) = \$50 \qquad\qquad \mathrm{Var}\ (\mu_X) = 20^2$$

$$E(\sigma_X{}^2) = 10^2 \qquad\qquad \mathrm{Var}\ (\sigma_X{}^2) = 32^2$$

Due to the high investment cost of each new press, the break-even monthly savings must be \$47,500 per unit. Production demands are estimated as steady throughout the planning horizon (no trend, seasonal, or cyclical components).

Cash flow control charts are prepared to monitor the project's performance. Since there are only four units, they will all be used to form a single grouping, and the group's monthly performance will be used as a sampling observation. The calculation of grouped performance beliefs is derived from the parent population estimates and equations (1) through (4):

$$E(\mu_{\bar{X}}) = E(\mu_X) = 50$$

$$\mathrm{Var}(\mu_{\bar{X}}) = (1/n)\ \mathrm{Var}(\mu_X) = 10^2$$

$$E(\sigma_{\bar{X}}{}^2) = (1/n)\ E(\sigma_X{}^2) = 5^2$$

$$\mathrm{Var}(\sigma_{\bar{X}}{}^2) = (1/n^2)\ \mathrm{Var}(\sigma_X{}^2) = 8^2$$

We desire $\pm 3\sigma_{\bar{X}}$ accuracy, and the CFCC for expected performance has the

following limits:

$$UCL_{\bar{X}} = 50 + (3)(5) = 65$$

$$Centerline_{\bar{X}} = 50$$

$$LCL_{\bar{X}} = 50 - (3)(5) = 35$$

The MRL is established (here, the parent population is normally distributed) from

$$P(x < 47.5) \quad = \quad P(Z < -0.25) \quad = \quad 0.4013 \quad = \quad P(x < \overline{MRL})$$

so,

$$MRL = (-0.25)(5) + 50 = 48.75$$

The CFCC for performance variability is constructed by using the standard coefficients for a subgrouping of size 4 ($d_2 = 2.059$, $D_1 = 0$, and $D_2 = 4.698$), such that its control limits are:

$$UCL_R = D_2\sigma_X = 4.698\ (10) = 46.98$$

$$Centerline_R = d_2\sigma_X = 2.059\ (10) = 20.59$$

$$LCL_R = D_1\sigma_X = 0\ (10) = 0$$

The assumptions that the marginal distributions of $\mu_{\bar{X}}$ and $\sigma_{\bar{X}}^2$ are normal and inverted-gamma, respectively, are considered valid, and the prior distribution parameters are found by solving equations (7) through (10):

$$m' = E(\mu_{\bar{X}}) = 50$$

$$E(\sigma_{\bar{X}}^2) = \frac{\delta' v'}{\delta'-2} = 5^2 \qquad \text{or,} \qquad v' = \frac{5^2(\delta'-2)}{\delta'}$$

$$Var(\sigma_{\bar{X}}^2) = \frac{2\delta'^2 v'^2}{(\delta'-2)^2(\delta'-4)} = 8^2$$

after substituting for v'^2, and cancelling terms

$$\frac{2(5^2)^2}{\delta'-4} = 8^2 \qquad \text{or,} \qquad \delta' = \frac{2(5^2)^2}{8^2} + 4 = 23.531$$

and,

$v' = 22.875$

$$Var(\mu_{\bar{x}}) = \frac{\delta' v'}{k'(\delta' - 2)} = 10^2$$

$k' = 0.3906$

The monthly savings generated by each forging press in the first 6 months of operation are listed in Table 1, along with the monthly average and range. The table values represent the discounted summations of costs and benefits incurred, carried from their time of occurrence to the end of the calender month. For brevity's sake, the summations and discounting are omitted.

To apply the techniques of this paper, the first month's observations are averaged (= 49.25) and then posted to the expected performance control chart. This value is within the chart's UCL and LCL, and is above the MRL, so, the first month's grouped mean appears consistent with the prior belief. On the performance variability control chart, the range of point one is 46.65, and it also falls within the chart's control limits. We then revise the prior parameters, using equations (11) through (14), as follows:

$$m'' = \frac{(0.3906)(50) + (1)(49.25)}{0.3906 + 1} = 49.46$$

$$k'' = 0.3906 + 1 = 1.3906$$

$$\delta'' = 23.531 + 1 = 24.531$$

$$v'' = \frac{(23.531)(22.875)+(0.3906)(50^2)+0+49.25^2-(1.3906)(49.46^2)}{24.531}$$

$$= 21.953$$

The revised estimates are found from equations (7) through (10):

Table 1. Discounted monthly savings for each forging unit, with group average and range.

Month	Observation				$\bar{x}$	R
	Unit 1	Unit 2	Unit 3	Unit 4		
1	32.37	79.02	42.04	43.56	49.25	46.65
2	10.95	45.51	57.61	38.37	38.11	46.66
3	58.96	56.06	45.73	35.66	49.10	23.30
4	53.43	27.61	54.16	7.19	35.60	46.97
5	63.91	39.84	33.97	44.33	45.51	29.95
6	55.48	28.59	39.21	47.70	42.74	26.89

$E(\mu_{\bar{x}}) = 49.46$

$Var(\mu_{\bar{x}}) = 4.15^2$

$E(\sigma_{\bar{x}}^2) = 4.89^2$

$Var(\sigma_{\bar{x}}^2) = 7.46^2$

These descriptive terms are listed as the second line of Table 2. In preparation for the second month's sampling, the control limits and MRL must be similarly revised, based on the new expected values for the subgroup mean and variance. The expected performance chart's revised control limits, MRL, and the probabilities that the subgroup will not meet the MRL, are listed in Table 3. The MRL has decreased slightly (48.75 to 48.48), but the probability of failing to meet the MRL has increased from 0.40 to 0.42. The second month's group average of 38.11 is within the control limits, but below the MRL. This is shown in Figure 2, along with the other monthly postings. It is important to note that by the time six monthly samplings have been completed, the MRL has an associated probability of failing to meet the MRL of 0.6327, a drastic increase from the original 0.4013.

An inspection of the monthly plottings (particulary of months 4 through 6) shows that the revised means are fairly accurately describing the cash flow. It is also apparent that the management should take interest in this project, because the MRL has been violated in 4 of the 6 months sampled.

For the performance variability control chart, Figure 3, the revised $E(\sigma_{\bar{x}}^2)_j$ generate new centerlines and UCL's for each month, and it appears that the sequential revisions are adjusting with the sample ranges. When the first month's sample is used to revise the UCL_R, the

Table 2. Revised estimates of distribution parameters.

Applicable Month	Prior Belief			
	$E(\mu_{\bar{x}})$	$(E(\sigma_{\bar{x}}^{-2}))^{\frac{1}{2}}$	$(Var(\mu_{\bar{x}}))^{\frac{1}{2}}$	$(Var(\sigma_{\bar{x}}^{-2}))^{\frac{1}{2}}$
1	50.	5.	10.	8.
2	49.46	4.89	4.15	7.46
3	44.71	5.11	3.30	7.94
4	46.01	5.06	2.75	7.61
5	43.64	5.28	2.52	8.11
6	43.98	5.19	2.23	7.68
7	43.79	5.09	2.02	7.27

Table 3. Expected performance control chart revised control limits, MRL, probabilities of failing to meet the MRL, and observed averages.

Month of Sampling	$LCL_{\bar{x}}$	Center-$line_{\bar{x}}$	$UCL_{\bar{x}}$	MRL	$P(\bar{x} < MRL)$	Observed Average
1	35.	50.	65.	48.75	0.4013	49.25
2	34.79	49.46	64.13	48.48	0.4207	38.11
3	29.38	44.71	60.04	46.11	0.6076	49.10
4	30.83	46.01	61.19	46.76	0.5584	35.60
5	27.80	43.64	59.48	45.57	0.6428	45.51
6	28.41	43.98	59.55	45.74	0.6327	42.74

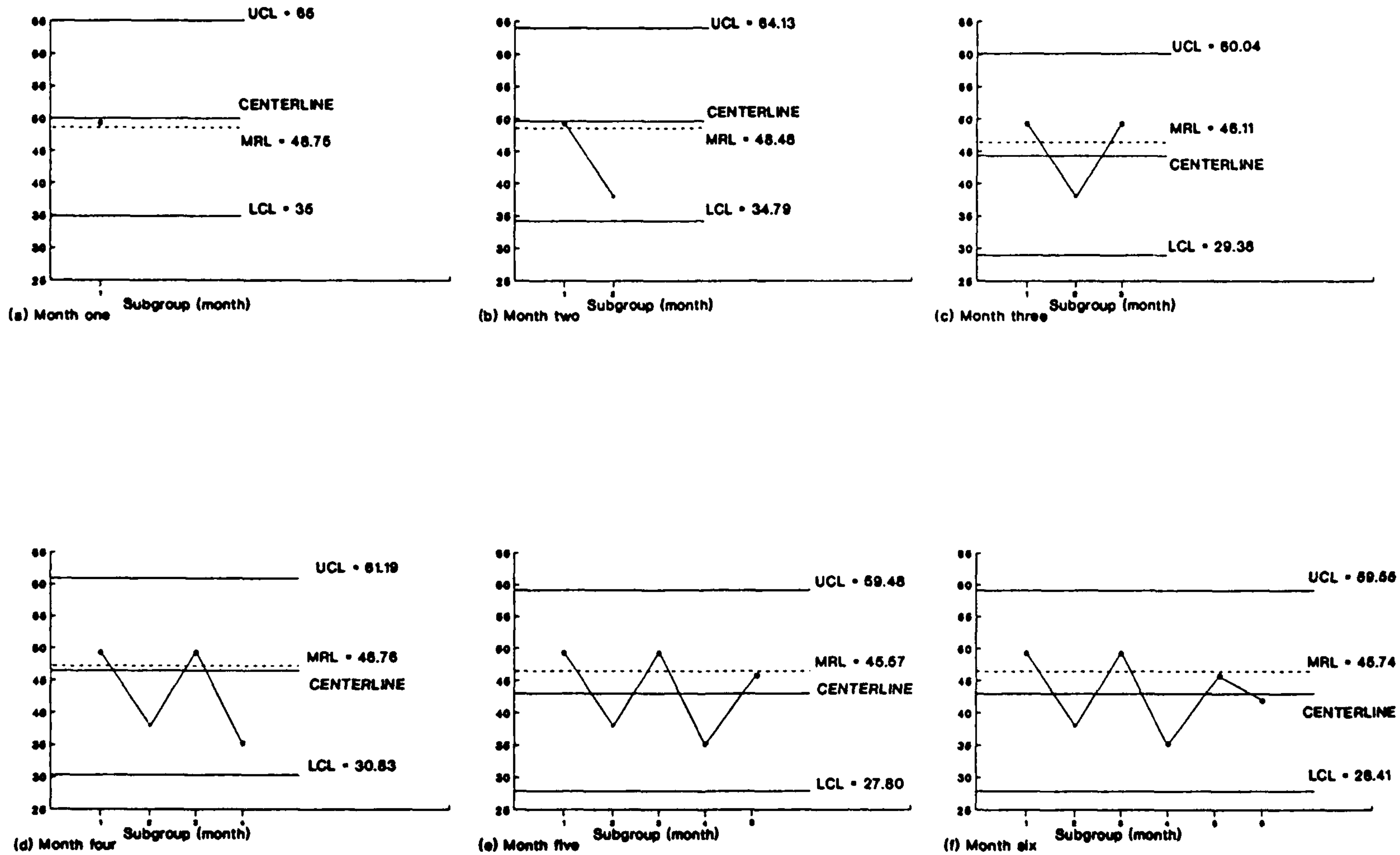


Figure 2. Monthly development of expected cash flow performance control charts.

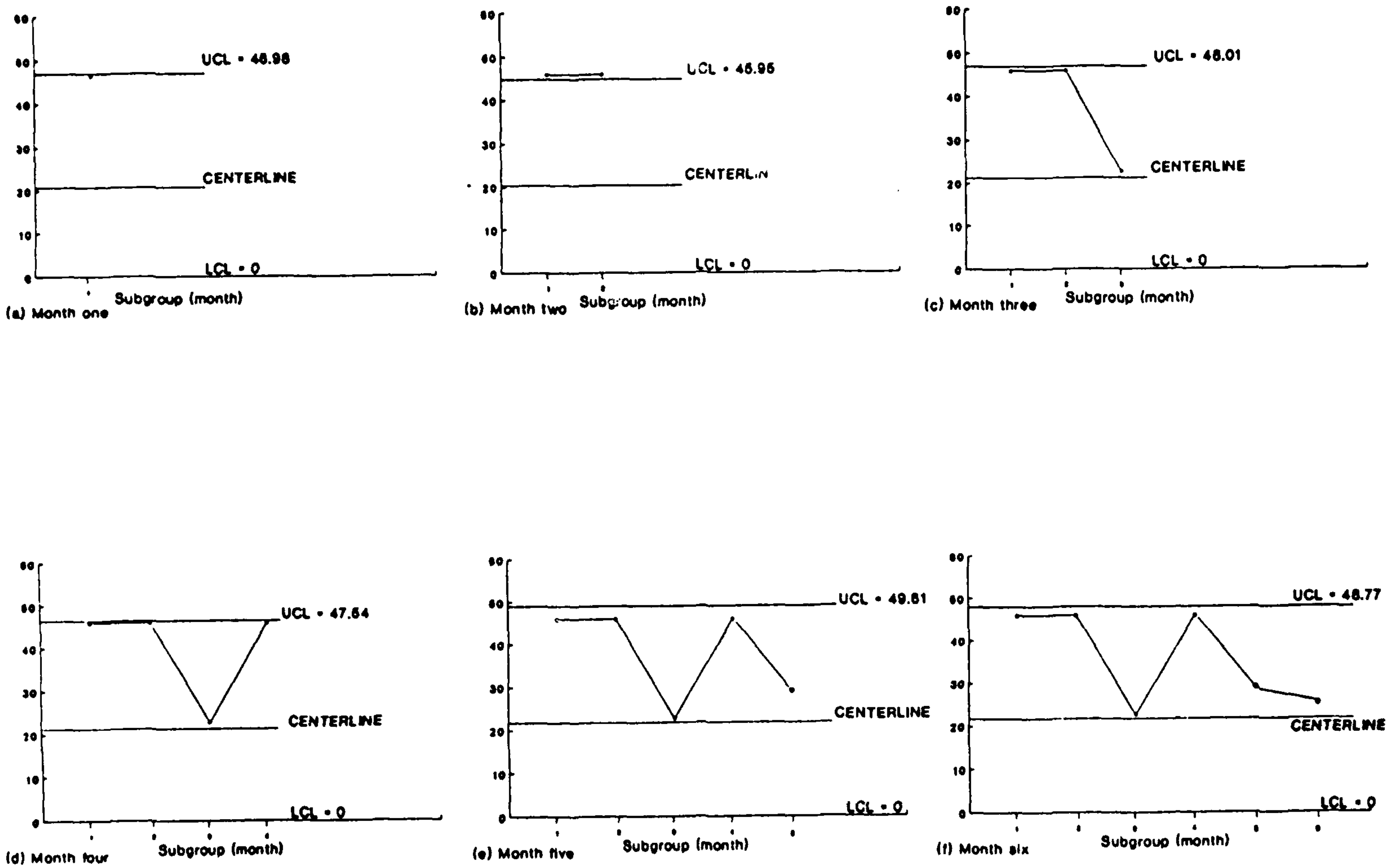


Figure 3. Monthly development of cash flow performance variability control charts.

point indicates an out-of-control condition. We interpret this as an indication that our initial estimate was somewhat inaccurate. When the second month's range is posted, it also indicates an out-of-control condition. However, when the second month's data is incorporated into the revised UCL_R (in preparation for month three), the first and second month's data points are no longer out-of-control, the revision process has adjusted the limits to compensate for those points. The monthly control limits are listed in Table 4. (For brevity's sake, we will omit the investigations of observation independence and normality.)

Analysis of Sample Data

The individual data points were generated from a $N(40, 20^2)$ distribution. For our subgrouping size of four, this corresponds to an average value distribution of $\bar{x} \approx N(40, 10^2)$. Clearly, the final version of the expected values for the grouped mean, 43.79, and variance, 5.09^2, are much closer than the original $(N(50, 5^2))$. The true $P(x < 47.5) = 0.6462$, is also closely approximated by the sample data (at 0.6327).

In comparison, the treatment of this data by conventional control chart methods has different results. Without our revision process, the control limits on each chart remain unchanged, as none of the samplings are out-of-control, and the run of six consecutive points below the mean does not constitute a revision requirement (seven are required). So, for the data generated, there is no reason to believe that the process is out-of-control. This belief is not as accurate as the revised belief. Also, since the minimum acceptable return that is not being met, this is a costly lack of information.

Table 4. Performance variability control chart revised limits and observed ranges.

Month of Sampling	LCL_R	Center-line$_R$	UCL_R	Observed R
1	0.	20.59	46.98	46.65
2	0.	20.14	45.95	46.66
3	0.	21.04	48.01	23.30
4	0.	20.84	47.54	46.97
5	0.	21.74	49.61	29.95
6	0.	21.37	48.77	26.89

One of the primary purposes of conventional quality control charts is to provide the operator with a clear picture of process performance. If there are trends, or out-of-control observations, those charts provide a visual aid to the operator. These cash flow control charts provide a visual aid, but in a distinctly different manner. As samples are collected and posted, any performance trends or abnormalities will still appear, but as the parameter estimates are revised, they will adjust so that they accurately describe those observations. The movement from out-of-control to in-control conditions is the visual depiction of uncertainty resolution.

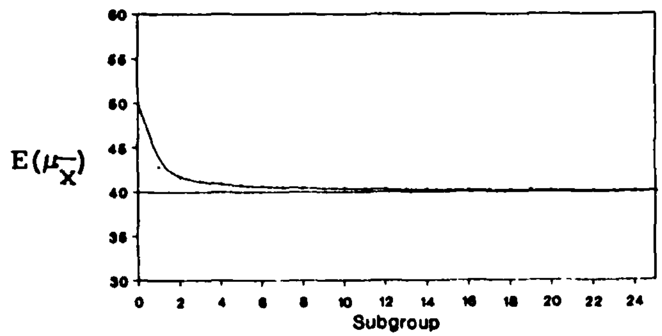
<u>Analysis with Theoretic Data</u>

As a further, more analytic demonstration of the system's revision capabilities, recall those initial prior beliefs

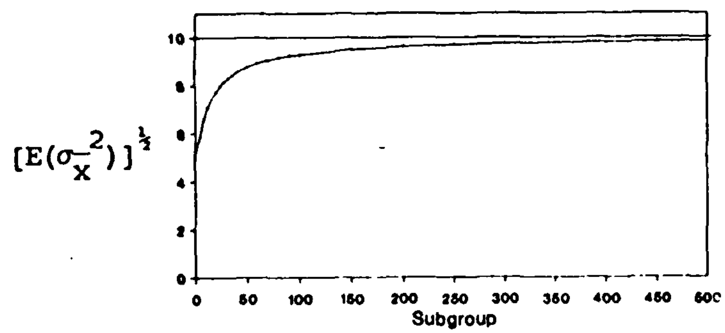
$$E(\mu_{\bar{x}}) = 50. \qquad Var(\mu_{\bar{x}}) = 10^2$$

$$E(\sigma_{\bar{x}}^2) = 5^2 \qquad Var(\sigma_{\bar{x}}^2) = 8^2$$

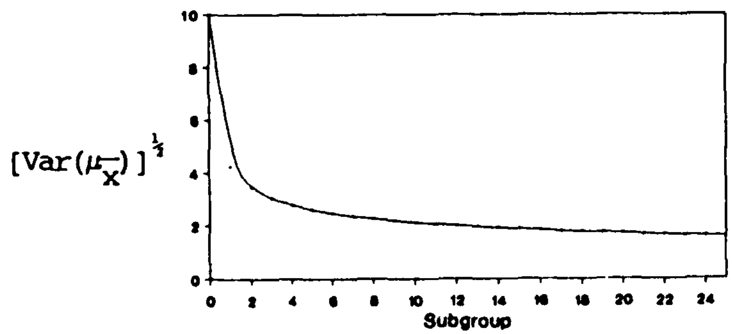
with the same initial values for parameters m', δ', v' and k'. The true but unknown, distribution of $\mu_{\bar{x}}$ is again N(40,10^2). But this time, every month's sample data is an exact representation of the true distribution, with m = 40, g = 1, v = 10^2. Repeated application of this data to the parameter revision formulas results with the expected value terms moving towards their true values, and the variance terms go to zero (the Var($\sigma_{\bar{x}}^2$) initially gets larger, peaking at sample 27, and then steadily decreasing). Figure 4 provides an illustration of how the expected values of the group mean and variance move towards their true values (in parts (a) and (b), respectively), and also how their



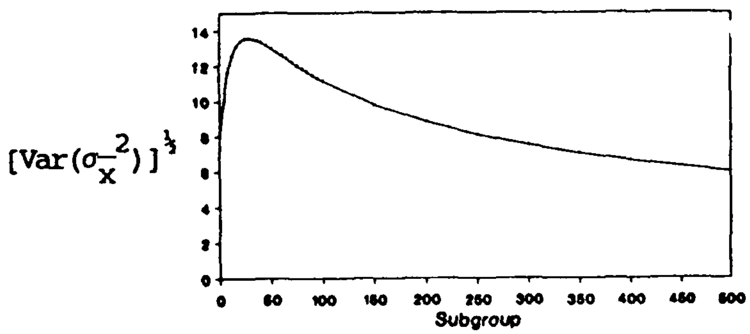
(a) Movement of the subgroups' mean from estimated to true value.



(b) Movement of the subgroups' variance from estimated to true value.



(c) Uncertainty resolution in the subgroups' mean value.



(d) Uncertainty resolution in the subgroups' variance.

Figure 4. Effects of Bayesian revision on parameter estimates, using representative samples.

respective uncertainties are resolved, as their variabilities go to zero (parts (c) and (d)).

CONCLUSIONS

The method presented here seeks to take advantage of the normal distribution properties of subgrouped data, and the relative stability of the cash flows for engineering replacement problems. Then, combine these properties with Bayesian revision techniques for the normal-gamma natural conjugate family of distributions and statistical quality control chart procedures, in order to provide a meaningful tool for economic project management and control. The example demonstrates that the revision process can clarify proposal estimation errors, and the chart construction process clearly provides a visual depiction of those revisions. The benefits, in more general terms, are that the uncertainty resolution provided by the revision process makes the current project a stronger incumbent in future comparisons, if its performance was underestimated, or a weaker incumbent if its performance was overestimated. Further, there are advantages to this method that are not available under conventional control chart methods. Application of this method can drastically affect the timeliness of decision strategy changes. It performs estimate revision before other procedures would do so, providing a means for near continuous updating. This type of flexibility provides improvement of information that can lead to economic gains for the firm. In the example problem, the revised process provided updating information five months before conventional control chart methods would have indicated any errors. In

summary, the near-costless information improvement that this system provides can be beneficial in a sequential decision making process.

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VII. CONCLUSIONS AND RECOMMENDATIONS

The purpose of this research has been to investigate the significance of using post-audit information in capital budgeting decisions. This research has focused on situations where capital projects have been implemented by companies that follow periodic decision review policies, and the sequential post-audit information of the implemented projects is readily available for each review. General areas of interest have been the development of a means to quantify the user's beliefs about the quality of the initial estimates, application of techniques to incorporate the post-audit information, and a means to incorporate the user's beliefs in the quality of the post-audit information. Additionally, the development of a management tool that captures uncertainty resolution is of interest. In the sections that follow, the results of this research will be summarized, followed by conclusions that can be drawn from those results, and recommendations for future research in this area.

Summary of Results

This research begins with a discussion of the problem elements that make manufacturing equipment replacement decisions one specific area of the various types of capital budgeting problems. A review of the literature reveals that an increasing number of companies are collecting post-audit information. However, the review also reveals

that only a limited amount of attention has been given to applying this information to capital budgeting, in general, with even less attention being given to equipment replacement, in particular. To address this shortfall, the following specific areas were investigated: (1) the adaptation of existing modeling techniques to provide the descriptions of the probability based estimates of the equipment cash flows, (2) the incorporation of the user's strength of belief in the quality of those cash flow estimates, (3) the use of post-audit cash flow realizations as sample information in the revision of the estimates, (4) the incorporation of the user's strength of belief in the quality, or repeatability, of the sample information, and (5) the development of a management tool to illustrate the uncertainty resolution that occurs in the estimates.

Adaptation of Modeling Techniques

The unknown nature of the future cash flows makes it appropriate to model them as probability functions. However, the level of detail incorporated into the estimates and the type and quantity of future sample observations makes it necessary to have several probability modeling techniques available. These methods involve using either discrete or continuous probability functions.

When the cash flow probability distribution is modeled with a discrete function, whether as the initial formulation or as an approximation to some continuous function, the formulation involves the assignment of probabilities to each outcome interval or category. Instead of simply modeling this situation with a discrete function, this research uses a Dirichlet distribution, which has descriptive

shape parameters that can be interpreted for the interval probabilities. Using this distribution, as opposed to a discrete function, creates no loss in descriptive quality.

When the prior distribution is to be modeled as a continuous distribution, this research uses the beta or normal distributions. The flexible nature of the beta distribution, through manipulation of the shape parameters α and β, makes it suitable for many prior probability distributions. This research uses the incomplete beta function as a means to refine the user's prior belief, thereby extending previous modeling efforts that limited themselves to using simultaneous solution of PERT and beta distribution equations. This refinement was keyed to the use of proportional changes in α and β. When the continuous belief is symmetric, this research examines the option of using the normal distribution. This procedure permits refinement of beliefs through manipulation of the distribution percentiles that are assigned to the extreme points of the three-point estimate.

Strength of Belief in Prior Estimates

Concurrent with the selection of a descriptive prior probability distribution was the development of a concept to incorporate the user's strength of belief in the quality of those initial estimates. This research examined several modeling techniques for each of the previously mentioned possible distributions.

When the initial beliefs are modeled with discrete intervals (whether as the initial belief or as an approximation to that belief), the Dirichlet distribution is used. This study introduces the concept of using the descriptive parameters to reflect the strength of belief

in the quality of the estimates. The term Information Quality Factor (IQF) was introduced as a relative measure of the quality of the initial estimates compared to the anticipated sample. The IQF was multiplied by the respective category probabilities to obtain the initial Dirichlet shape parameters. The stronger the belief in the prior estimates, the larger the values assigned to the shape parameters, α_j's.

When the initial beliefs are modeled as continuous probability distributions (beta or normal), the concept of the user's strength of belief was similarly incorporated. The previously mentioned manipulation of proportional α and β shape parameters serves a dual purpose. Because the shape parameters, α and β, are transformed into the terms, r' and n', for revision by sample information, the use of proportionately larger α and β values has the effect of creating a larger value of n'. This effectively reflects a stronger belief than lower shape parameter values. This concept was also extended to normal prior beliefs through the use of the equivalent sample size and the term n'. As a direct result of this concept, this research develops a method to approximate the population standard deviation, for use in an unknown mean, known variance (assumed) situation. (This method is used in situations when there is insufficient information in the project proposal for an unknown mean and variance approach.)

Estimate Revision

The development of the prior probability distributions had the immediate concerns for description and strength of beliefs. The subsequent task, but one that was considered simultaneously during the

development, is efficient revision of those prior beliefs through the use of post-audit sample information. This research uses existing Bayesian techniques to perform the revisions, and while the techniques are not new, how they are applied to the replacement situations are innovations.

When the prior beliefs were described as a discrete model, this research uses a Dirichlet probability distribution. This distribution was selected in anticipation of the impending sample data. The critical procedure in this modeling approach is to manipulate the modeling of the cash flow realizations as categories, or intervals, that correspond to the intervals in the prior belief. This gives the sample results multinomial distribution characteristics. More importantly, the sample and prior belief create a Dirichlet-multinomial conjugate distribution, which makes the prior distribution naturally responsive to the categorized sample information.

When the prior belief is modeled as a beta distribution, it is necessary to transform the sample data from the form used in the preposterior analysis, to the form that is used for the revision process and posterior analysis. During the preposterior analysis, the sample data is predicted as either being above or below its expected value. This gives the sample data a binomial characteristic, and sample outcome projections are made from the use of the beta-binomial predictive distribution. The sample data is collected on a continuous (approximately) monetary index. This research transforms that scale to an equivalent success/failure index, through the terms r_e and n, or n_e (more on the term n_e is covered in the immediately following section),

to regain the binomial characteristics. This transformation determines the number of "successes", r_e, to be the proportion of the outcome range attained by the sample observations, n, or equivalent samples, n_e. This definition permits the prior belief to be revised by the sample information, through the use of the beta-binomial natural conjugate distribution. When the prior belief is modeled as a normal distribution, the procedures (with the exception in the section that follows) follow conventional Bayesian procedures.

Strength of Belief in Sample Quality

Two facts precipitated the need for the user to be able to incorporate his strength of beliefs in the quality of the cash flow realizations, or sample observations: (1) the installation and implementation of the equipment may not occur in the same fashion that was projected at time zero, and/or (2) the financial conditions outside the factory may not be as anticipated. In either case, the changes may alter the perceived quality of the initial estimates. In each of the distributions developed in this study, the strength of the prior beliefs was based on a relative comparison of their perceived quality and an anticipated number of sample results. These results are to be taken under an assumed set of circumstances. How accurately the actual conditions compare to those assumptions determines the magnitude of the equivalent sample size adjustment, which is essentially the application of a multiple, or scaling factor, to the observed results ($n_e = k * n$, where k is the magnitude of the scaling factor).

The equivalent sample size, n_e, is designed to reduce the quantity of the sample observations when the conditions are anomalous

and not likely to occur again. The equivalent sample size increases the quantity when the conditions are anomalous, but are considered to be more representative than those used for the initial estimates. When the conditions are as projected, there is no change.

Illustration of Uncertainty Resolution

As cash flow realizations replace estimated values, uncertainty resolution occurs. This research investigated the concept of modifying statistical quality control charts (specifically, $\bar{x}$ and R-charts) to graphically portray this resolution. The situation examined was an equipment fleet replacement problem, with the population of the individual unit's cash flows assumed to be normally distributed with unknown mean and variance (this situation was selected to take advantage of the Central Limit Theorem's properties). This procedure utilized the normal-gamma family of conjugate distributions. These procedures found that as uncertainty resolution occurred, the upper and lower control limits would adjust towards the true values, providing an illustration of the resolution. The use of the Minimum Return Limit permits the user to see how the resolution is affecting the anticipated profitability of the project.

The study examined the behavior of the uncertainty resolution when repeated representative samples were used to revise the initial estimates. This found that the expected values for the unknown mean and variance move quickly to their true values. However, the variance of the expected value of the mean, a measure of uncertainty resolution, moved slower to zero, or complete resolution. Further, the variance of

the expected variance moved at an even slower rate to complete resolution.

Conclusions

This research shows that sequential post-audit information, in the form of cash flow realizations, can be used to provide uncertainty resolution in implemented capital projects. This research presents several probability based modeling techniques that can be used to illustrate varying amounts of uncertainty that accompany initial cash flow estimates. The flexible natures of the beta and Dirichlet distributions make them appropriate for many problem situations. An assessment of the quantity and quality of information that was available for the formulation of those probabilistic estimates can be incorporated into the initial model, as a reflection of the user's strength of belief in those estimates. Similarly, the conditions surrounding the occurrence of particular cash flow realizations can provide input to the user in his determination of the replicability of those results. Once assessed, this information is incorporated into the revision process. The impacts of these decision modeling techniques can affect the previously determined equipment implementation strategies, creating the need for company decision strategy adjustments. These adjustments may include changing the timing and/or amounts of future investments (the basis of the equipment replacement problem). On the other hand, the uncertainty resolution may not create decision strategy changes if events occur as anticipated, but the resolution still improves the quality of the initial estimates. This information gain can then be used in future

comparisons of the defender asset with new, currently unknown, challenger assets. The incorporation of just the user's strength of belief in the prior information or just the sample information can provide clarity to the decision problem.

The use of the cash flow control charts provides a graphic illustration of the uncertainty resolution process, and provides the user with an easily identifiable indicator (MRL) that can be used to initiate the decision strategy changes. The proximity of the expected value center line and MRL graphically depicts the discounted "break even" performance of the project.

These concepts, and the potential gains available through decision strategy adjustments, will improve a company's overall investment performance. However, the potential benefits of these decision-aiding concepts must be considered with their potential abuses. These measures are designed to incorporate the decision-makers unbiased beliefs in the quality of information. If these methods are applied in a parochial manner, the results will be similarly biased decision strategies.

Recommendations for Further Research

A logical extension of this research would be an investigation of how these post-audit information concepts, and the uncertainty resolution process in general, are affected when the initial estimates include trend, seasonal, and/or cyclical components that have uncertainties associated with their respective estimates. This task may require the development of a joint factor resolution model.

Another area would be a further investigation of cash flow control charts. As many cash flows are uneven, the use of CFCC based on residuals could be investigated, as well as an investigation of the use of CUSUM charts, to detect small shifts in trend. Another area could be the development of CFCC when the Central Limit Theorem is not appropriate. The procedures developed in this study were modifications of methods for $\bar{x}$ and R-charts. The investigation in this area may involve using other statistical quality control chart methods, with some transformation of the reported cash flow data.

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